CALC CONF 3

UNIVERSITÀ DEGLI STUDI DI MILANO

# The Learning and Teaching of Calculus across Disciplines 2

## - Pre-Conference Proceedings -

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MATH EXCELLENCE PROJECT

UNIVERSITÀ DEGLI STUDI DI MILANO
DIPARTIMENTO DI MATEMATICA
"FEDERIGO ENRIQUES"

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# Papers

# Accumulation as a tool towards blending reasoning about quantity and rate of change in physics contexts

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*The amount, change, rate and accumulation of physical quantities are essential features of reasoning with calculus and physics. Experts in physics and mathematics use rate of change reasoning throughout their process of developing and making sense of graphical models; distinguishing between rate and quantity is an essential part of that. We suggest that rate vs time graphs offer an opportunity for direct instruction on distinguishing between rate and quantity, as well as blending this reasoning to determine an accumulation. Here we share some pilot-tested graphical reasoning activities that we have developed based on the ways experts and students reason.*

*Keywords: covariation, graphical reasoning, quantity, physics*

## INTRODUCTION

Making sense of quantity, rate of change, and accumulation are central features of calculus (Carlson et al., 2002; Samuels, 2022). Research in mathematics education and physics education has demonstrated that distinguishing between quantity, rate of change, and accumulation is difficult for students (Sealey, 2014; Trowbridge & McDermott, 1980; Von Korff & Rebello, 2012; Yu, 2024). Research has also demonstrated that physics experts distinguish between rate and quantity in part by identifying physically meaningful points in graphical representations and reasoning about the rate of change around those points (Zimmerman et al., 2023). One possible way that calculus and physics instructors may be able to help their students learn to think this way is by using direct instruction of these expert-like behaviours.

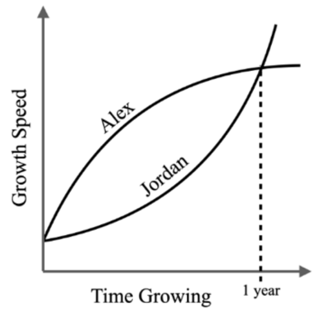
Graphical representations with meaningful accumulated *physical* quantities typically involve a rate of change represented on the vertical axis, and time or position on the horizontal axis. In physical contexts, it is also common that the rate of change is a quantity in its own right (e.g. speed is the time rate of change of position, the accumulated quantity in a graph of speed vs time is a displacement). Reasoning about accumulated quantities using graphs of rate vs. time therefore requires students to be able to identify the physical quantity represented by the vertical axis as a rate, interpret the meaning of its rate of change, and use both pieces of information to determine the accumulation as a distinct quantity. Rate vs. time graphs thus provide a rich representation that blends several ways of reasoning about quantity and rate of change, that are ubiquitous in physics courses.

## STUDENT REASONING AROUND ACCULUMATION TASKS

The item shown in Figure 1 is one example, featuring a rate (growth speed) vs time graph and asking students to reason about an accumulated physical quantity (amount of growth in 1 year). This task is derived from a survey that assesses physics quantitative literacy (White Brahmia et al., 2021; Zimmerman et al., 2022). The data we share come from a series of 29 individual student interviews conducted as part of validating the inventory.

Students were solicited for interviews from an algebra-based introductory physics class at a large U.S. university. The course is typically taken by 3rd and 4th year university students studying life science, most of whom have completed at least one semester of calculus instruction that includes basic integration. Interviews involved one student and one member of the research team; student participants were asked to work through the items while talking out loud. Interviews were audio recorded and transcribed. Students were offered $15 gift cards as a small thank you for their time. We do not claim that these ways of reasoning are representative of all physics students; rather, we share this evidence to illustrate the varied mental resources these students activated about physical quantities and rates of change at the beginning of an introductory physics class after having taken calculus.

The graph at right represents how fast two children are growing vs time. The children are named Alex and Jordan, and their growth is measured starting on their 10th birthday. Which of the following choices best describes how much the children have grown in one year?




a. Alex and Jordan have grown the same amount.

b. Alex has grown more than Jordan.

c. Jordan has grown more than Alex.

d. The graph does not provide enough information to compare how much the two children have grown.

**Figure 1: An example of an accumulation item.**

Most of students that we interviewed chose answer options (a) or (b). Students answers and justifications are shown in Table 1. Students who chose (a) often did so either because they conflated the quantity "growth amount" with the quantity "growth speed", or because they examined the *average* growth speed which is the same for both children. Students who chose (b) did so either by noticing that the accumulated quantity (how much the children grow) can be found by taking an integral, or by reasoning that Alex's growth speed is larger than Jordan's the entire time.

There are multiple interpretations for the students who used an intersection approach. One could reason they were distracted by the intersection and viewed the vertical axis as representing total growth. However, this student previously articulated that they understood the vertical axis represented growth rate. Another interpretation is that this student conflated quantity and its rate of change while trying to use them together to find the total growth. Students who chose (a) and discussed average rates of change represent an opportunity for direct instruction. These students have productive quantitative resources around accumulation with linear functions that can be built upon, but do not yet have facility with non-linear changing rates of change.

We recognize that students who chose (b) and understand the procedure for taking an integral may or may not have strong conceptual reasoning around accumulation. We suggest that students who chose (b) and spontaneously chose to compare the changing rates of change demonstrate strong conceptual reasoning about accumulation for quantities with changing rates of change.

| Answer Choice | Approach | Example Quote |
|---|---|---|
| a | Intersection | "They intersect right here, despite having two different curves for their growth. So that means despite their different rates of growth at this specific year, they [have] grown the same amount." |
| a | Average Rate of Change | "Since they both have the same growth speed at the end of the year, they have grown the same amount? Because... they have like the same average speed." |
| b | Area Under Curve | "So I'm thinking that it's like a physics problem where it's like the area underneath the graph. That would mean Alex grew more than Jordan." |
| b | Relative Value of Rate of Change | "Alex, their, like, their growth speed is just higher for more of the year. So they're just gonna grow more." |

**Table 1: Common student approaches to the item shown in Figure 1.**

These data suggest that problems that ask students to reason about the accumulated quantity represented in rate vs time (or position) graphs may be a fruitful place to help them learn to differentiate between quantity, rate, and accumulation and to better understand how these three kinds of quantities are related.

## EXAMPLES OF ACCUMULATION ACTIVITIES

We designed activities in the context of a large-enrollment (N = 323) algebra-based Introductory Electricity and Magnetism course to support students learning: (1) to distinguish between quantity and rate, (2) to reason about changing-rates-of-change rather than an average, and (3) reasoning that blends procedural and conceptual competency with rates of change, independent of calculus algorithms. We note that deciding whether to treat a physical quantity as a rate, quantity, or accumulation in a particular context is one part of "learning to distinguish" between them. To facilitate variation between instructors' instructional preferences, the activities were designed to be administered as clicker-questions during lecture or as practice exam questions. We also included a small number of these items as exam questions as an early measure of whether student reasoning was improving. These items represent our initial pilot into whether accumulation-based activities may help students learn to reason this way.

An example is shown in Figure 2, in which students compare how much heat is transferred across two rods. They are given a graph of $P$, the rate at which heat is transferred, vs $t$, elapsed time. The rate at which heat is transferred can be thought of as the amount of heat that moves from one end of the rod to the other in each time unit.

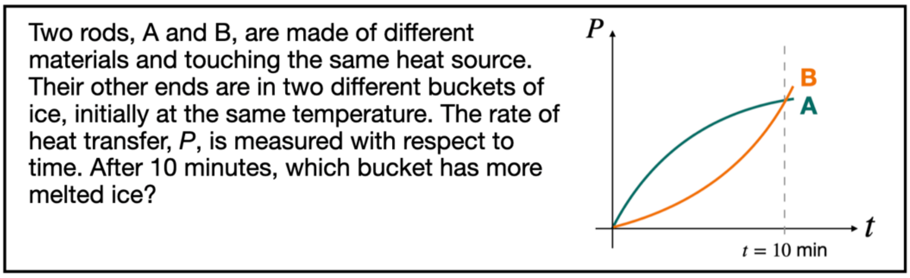



**Figure 2: An example of an in-class accumulation activity using a rate vs time graph.**

The mid-term exam questions (Fig. 3) provide an early measure of how students' reasoning improved. Although they are mathematically analogous items, they are not rigorous measures of what students learned in the course. Students likely have more facility with some physics contexts (metabolic energy) than others (electric circuits). However, these results paint a picture of how challenging, and context dependent, this kind of reasoning can be for students in a science course–even for students who have completed one or more semesters of calculus. 37% of our students chose the correct answer on the first midterm item, and 57% of our students chose the correct answer on the second. We note that both current and power were directly taught as rate quantities.

We interpret these data as an illustration that students require significantly more opportunities to practice with accumulation than we were able to offer in our preliminary pilot, or than they are getting in their calculus and physics classes alone.

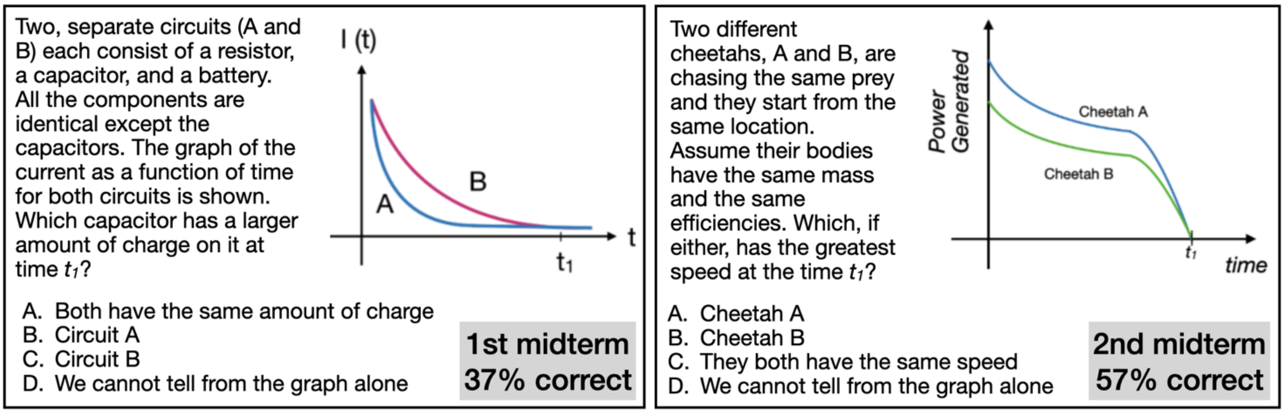



**Figure 3: Example exam items from an Introductory to Electricity and Magnetism course (N = 323) The left was given on the first midterm, and the right on the second.**

## INSTRUCTIONAL IMPLICATIONS

One benefit of incorporating graphical tasks alongside symbolic ones is that there is a high level of conceptual calculus-like reasoning without requiring a high level of procedural proficiency. In introductory physics classes, proficiency with symbolic reasoning is often not consistent across students. It is also typical in physics for graphical questions to act as practice after symbolic ones, despite research that has demonstrated the benefit of a multiple representations approach (Kohl & Finkelstein, 2008). By offering these activities alongside symbolic problems students were grappling with, we leveraged graphical reasoning from the very beginning of the unit.

Our study suggests that university students who have completed calculus and introductory physics are not likely to have strong proficiency with the foundational mathematical ideas of quantity, rates and accumulation. We suggest that these ideas are complex and take time to learn; likely more than any one term university course can manage. Incorporating instruction about accumulation in graphical contexts in calculus courses and across math-based STEM disciplines, that has a common focus and common language, can help students when using calculus to model physical phenomena. We present this work to help foster rich collaboration across disciplines.

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# An Analysis of Covariational Reasoning for the Conceptual Introduction of Derivative in US and Chinese Calculus Textbooks

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*Content analysis methods were used to examine the development of covariational reasoning levels in four calculus textbooks. One calculus textbook from each category was investigated: US college, US high school, Chinese college, and Chinese high school. The sections that conceptually introduce derivatives were selected for analysis. Conceptual analysis revealed that although none of the textbooks in this study have a coherent, systematic development of the pedagogy in covariational reasoning, the Chinese high school text provides the most development at intermediate levels. The relational analysis revealed that the US college text provides an abundant scaffolding for transitioning between the average and instantaneous rate of change, while all others lack transitions among passages that stimulate covariational reasoning.*

*Keywords: Derivative, Covariational Reasoning, Content Analysis.*

## INTRODUCTION AND REVIEW

Covariational reasoning, a developmental capacity to coordinate two covarying quantities, is essential in understanding and modeling dynamic events (Carlson et al., 2002; Thompson & Carlson, 2017). It is indispensable in physics (Sokolowski, 2021), chemistry (Rodriguez et al., 2019), biology (Bennoun et al., 2023), earth science (Gonzalez, 2022), engineering (Nunez et al., 2021), and economics (Mkhatshwa, 2024). Moreover, the developmental process of covariational reasoning capacity should not be viewed as an intermediate step that can be discarded once students recognize that the concept of the derivative is the instantaneous rate of change; as Bennoun et al. (2023) showed, the modeling of a dynamic event in biology can be more productive by going thoroughly through some “change equations” without jumping directly to the notion of derivative. Thompson and Carlson (2017) stated, "the meanings of calculus that are grounded in covariational reasoning also fit precisely with the ways of thinking that science educators complain is lacking in their students' mathematics" (p. 453). For calculus teaching, the challenge is how to emphasize covariational reasoning in the calculus curriculum (Thompson & Carlson, 2017).

Mathematics textbooks can heavily influence students’ learning outcomes by shaping students’ opportunities to learn and influencing the quality of instruction (Carroll, 1963). Recent research and analysis of calculus textbooks and curriculum (e.g., Tallman et al., 2021; Toh, 2021) found that in general, "students are rarely required to interpret functions or situations in terms of covariational reasoning such as coordinating changes in output for successive equal changes in input" (Tallman et al., 2021, p. 582), and syllabus and the textbooks only discuss the shape of the graph, without mentioning how the shape of the graph is related to the change of one variable when the other changes (Toh, 2022). Understanding how current calculus textbooks guide students in developing covariational thinking capabilities can benefit many STEM education stakeholders.

The research question of this study is: How is covariational reasoning developed in four calculus textbooks from the reader-center's perspective, one each in the following categories US college, US high school, China college, and China high school, and what are the similarities and differences?

## THEORETICAL FRAMEWORKS

This study rests upon two frameworks. The first is the five-level framework by Carlson et al. (2001) and its updated version (Thompson & Carlson, 2017), which is used as the base for the covariational reasoning developmental process. The other one is the framework of reader-oriented theory (Weinberg & Wiesner, 2011), which informs the research design in terms of coder selection, coding rubric development, and coder training. The content analysis of the textbooks in this study consisted of conceptual and relational analysis. Conceptual analysis identified the occurrence of texts that explicitly promote each of the five levels of covariational reasoning. Relational analysis identified the relationship among the occurrences of these passages.

## METHODOLOGY

The study underwent three phases: textbook and section selection, conceptual analysis, and relational analysis. In the first phase, I selected four calculus textbooks, shown in Table 1, all widely used in their categories. I identified the sections that conceptually introduced derivatives in each of the four textbooks. The derivative is the topic all calculus textbooks cover, and all five levels of covariational reasoning can be used to build up the mental image of changes. In the second phase, I worked with two coders who independently conducted a conceptual analysis of the selected sections. Both coders were fluent in both Chinese and English, had experience in both high school and college teaching in the US, were educated in both China and the US, and were trained by the author to conduct conceptual analysis for covariational reasoning (Chen, 2023). The process of conceptual analysis went through several iterations to ensure the result was independent and reliable (Chen, 2023). Each sentence that promotes a certain level of covariational thinking was identified, named L1 (covariation of variables is explicitly written), L2 (the direction of the change of one variable is explicitly written with the change of another variable), L3 (the amount of the change of one variable is explicitly written with the change of another variable), L4 (the average rate of the change of one variable is explicitly written with the change of another variable), and L5 (the instantaneous rate of change of one variable is explicitly written with the instantaneous change of another variable). The relational analysis in the third phase determines how L1 through L5 are related. Three levels of connection were identified: none, simple, and strong. Simple connections are connections between concepts, such as the use of the average rate of change after it is defined is a simple connection with the original definition. A strong connection places the simple connection within the same context, such as the same example or exercise.

## RESEARCH FINDINGS

Table 1 shows an overview of the sections from all four calculus textbooks.

| Category | US College | US HS | China College | China HS |
|---|---|---|---|---|
| Textbook | Stewart (2016) | Finney et al. (2016) | Higher Mathematics (2014) | Electives 2-2 (2012) |
| Section Pages | 105-117 | 87-96 | 73-84 | 1-11 |
| Main text | 118 | 92 | 113 | 77 |
| Exercises | 61 | 71 | 20 | 12 |

**Table 1: Overview of sections in this study's four selected calculus textbooks**

After each occurrence of the sentence that promotes various levels of covariational reasoning (T1 to T5) was identified, the progress of the levels through the whole section was investigated. Table 2 shows the occurrence and development of passages that explicitly promote covariational reasoning in the sections that conceptually introduce derivatives in all four calculus textbooks in this study.

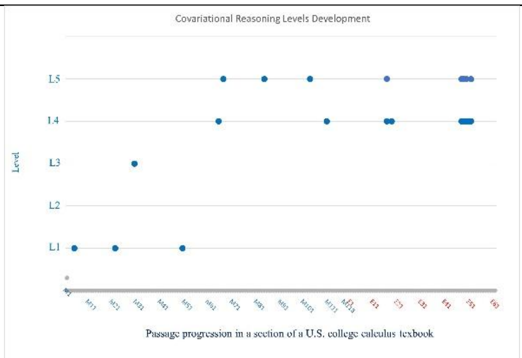
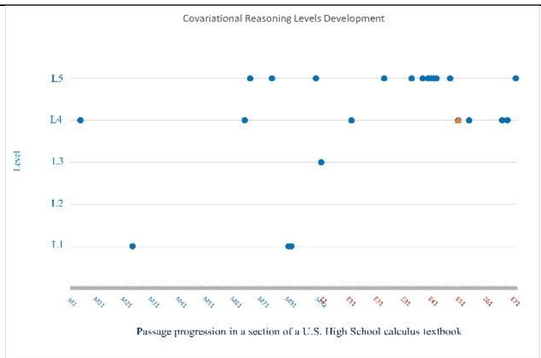
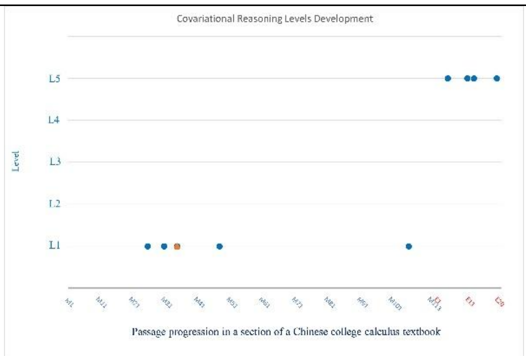
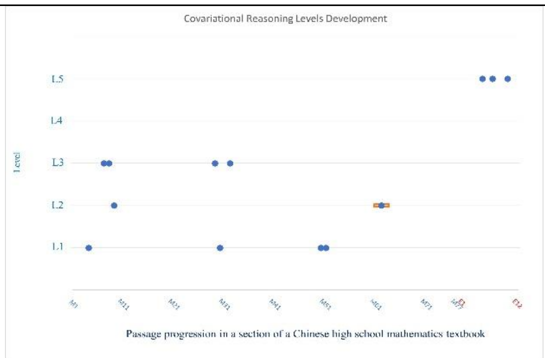
| | College | High School |
|---|---|---|
| The US |  |  |
| China |  |  |

**Table 2: The occurrence and the development of passages that explicitly promote covariational reasoning in the four calculus textbooks in this study**

All four textbooks have some occurrence of L1 and L5. Second, what happens between the starting and ending points differs from textbook to textbook. The Chinese college calculus textbook had no stimulants to covariational thinking in between, while the Chinese high school calculus textbook had the most elaborate and robust buildup of L2 and L3 before it got to L5. Third, neither Chinese textbook had developed L4 and L5 in the main expository text. The L5 only exists in the exercise sections of both Chinese textbooks. Fourth, both college calculus textbooks develop the covariational reasoning level mainly in one direction, i.e., the level goes up as the passage progresses. On the contrary, in the main expository section, both high school calculus textbooks developed the covariational levels in a circulated way. Fifth, both US calculus textbooks are heavy in L4 and L5. Sixth, only the US college textbook makes meaningful and substantial efforts regarding connections among concepts. However, scaffolding only happens among L4s and L5s. No meaningful transition, among other levels of covariational thinking, was found in the sections selected for this study. Sixth, one unique feature of the Chinese high school text is its substantial effort in building L2 and L3.

## CONCLUSION

Calculus education should emphasize the development of covariational reasoning, which is indispensable in many disciplines (Sokolowski, 2021; Rodriguez et al., 2019; Bennoun et al., 2023; Gonzalez, 2022; Nunez et al., 2021; Mkhatshwa, 2024). In addition to serving as a scaffold for developing derivative concepts, the intermediate level of covariational reasoning, the direction, magnitude, and the average rate of change are essential to mathematizing real-world dynamic events in writing the "change equations." In analyzing selected calculus textbooks in four categories, it was found that all four calculus textbooks did not make the developmental process of covariational reasoning explicit. This study suggests that the developmental effort in covariational reasoning pedagogy is one possible area for future improvement in calculus textbooks, whereby authors can explicitly consider scaffolding and advancement in students' covariation reasoning capacity.

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# An unexpected construct toward the fundamental theorem

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*Using a graphically based activity situated in the extra-mathematical context of water flowing into a pool, high school students were guided to construct the accumulation function for different flow rates. The analysis of the knowledge construction processes, using the theoretical framework of Abstraction in Context, revealed an unexpected construct: some students identified the flow rate as the derivative of the accumulation function. The analysis highlights the role of the context and indicates elements of knowledge that may be crucial when introducing integration.*

*Keywords: Integration, accumulative thinking, construction of knowledge, unexpected construct, Fundamental Theorem of Calculus.*

## INTRODUCTION

Most high school calculus students who will study calculus at the tertiary level will do so to use it in their natural or social sciences major. Experiencing the fundamental concepts of calculus: derivatives, integrals, and the fundamental theorem (FTC) in suitable extra-mathematical contexts (EMCs) already at high school, may support students' conceptualization of these concepts as well as their motivation to learn calculus. In this paper, we present evidence for the support of conceptualization by using EMCs (Gravemeijer and Doorman, 1999; Reinke, 2020).

We use a learning activity which aims to serve as an introduction to integration via accumulation in an EMC. We analyse students' learning process using Abstraction in Context as theoretical framework. Our results show that some students make use of the EMC to conceptualize even beyond what was intended by the design of the activity.

## THE LEARNING ACTIVITY

The learning activity is graphically based and designed to offer students an opportunity to develop Accumulative Thinking (Falach et al., 2025) in the context of water flowing into a pool. We define Accumulative Thinking as awareness of and ways of reasoning with the nature and the multiplicative structure of the "bits" that accumulate as well as with the dynamism of the process of accumulation.

We adopted accumulation from rate (Jones & Ely, 2022) as didactical basis for learning about accumulation. The activity has three parts, each using a graphically presented rate of flow of water into the pool: (1) constant (2) constant in segments and (3) linear and decreasing. The learning activity typically carried out in two 70-minute sessions. In each part, students construct a function representing accumulated water over time by analyzing the accumulating "bits" as products of time × rate, examining how the rate affects them, and summing these bits to form the accumulation function.

The third part of the learning activity is central to this paper. In this part, the students are working with a positive linear decreasing rate of water flow (Figure 1).

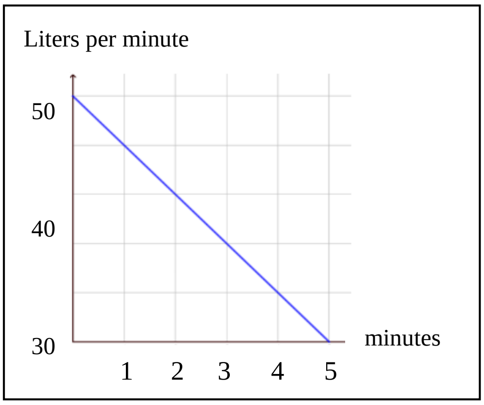


**Figure 1: Rate of water flow into the pool in part 3 of the activity**

Three pairs of grade 11 students were observed and audio-recorded during the activity. These students had learned about differentiation in grade 10 but had not yet been taught integration. Their work has been transcribed and analysed (Falach et al., 2025).

## THEORETICAL FRAMEWORK

We adopted Abstraction in Context (Dreyfus et al., 2015) as theoretical framework. It provides analytic tools to study how students construct mathematical knowledge, through an a priori analysis of the activity, yielding a structured list of intended knowledge elements; and an a posteriori analysis that reveals both intended and unexpected constructs. To describe students’ epistemic actions during abstraction, we used the RBC model (Recognizing, Building-with, Constructing), which characterizes abstraction as a process in which students identify relevant prior knowledge, use it in reasoning, and reorganize it to form new constructs.

The learning activity leads the students to draw the accumulation function by considering the bits that accumulate, the way they accumulate and the graph of the accumulation function. For the constant (part 1) and constant in segments (part 2) rates of flow, the students can calculate the accumulated amounts in consecutive time intervals by multiplying the rate by the time duration. In part 3, this is not applicable since the rate is linear and decreasing rather than constant. The area under the graph as representative of the accumulated amount can bypass this difficulty; therefore, students are led to construct this knowledge prior to the third part of he activity.

The design avoided referring to differentiation to keep students focus on accumulation and its features. Nevertheless, some students recognized that the rate graph represents the derivative of the accumulation function. The following sections describe the process that may have led to this generalization.

## THE CASE OF ANA AND ZOE

When considering the constant water flow rate in part 1 of the activity, Ana said “the filling rate will always be 30 litres per minute”; she recognized that the rate would impact the accumulation function in a predictable manner, thus laying a foundation for her understanding of how rate of change and accumulation are related. When asked to compute and graph the amounts of water accumulated in successive time intervals, Ana noted that the slope of the accumulation function reflects this constant rate and checked her observation by computing the algebraic expression $y = 30t$ by using two points. She concluded "the slope is constant, and it equals 30 because that's the rate… [the accumulation function] is a linear function starting at 0, with a slope of 30".

Next, when presented with a GeoGebra animation, in which a constant graph represents the rate of water flow as a function of time and the rectangular area under the graph is being filled in from the starting time and grows continuously to the end time, Ana reinforced her idea that the rate of flow was tied to the accumulation function. Ana realized that the product of rate and time interval gives the amount added and used this knowledge to explain why the area of a rectangular bit represents the amount of water added, a construct she used in the following tasks.

When dealing with the case of a rate that is constant in segments (part 2), Ana expressed the connection between the flow rate and the slope: " …we talked about it earlier that [the flow rate] is the slope, the litres per minute represent the slope [of the accumulation function]". To draw the accumulation function, Ana finds the area under the graph by multiplying the rate with the length of the time interval in each segment and summing them up to find points on the graph of the accumulation function.

Finally, when dealing with the case of a positive, linear, decreasing rate in part 3 of the activity (Figure 1), Zoe suggested using the area under the graph. However, Ana answered: "if this [graph] represents the slopes, then the derivative also represents the slope at any point". She graphed the accumulation function by plotting 3 points and connecting them by a concave down curve. She explained the graph by writing:

> If I know that this [rate] graph describes the slope at that point in the graph of the accumulation function, then I can determine that it is its derivative and then go from the filling rate graph equation, which is the derivative of the original function that is the accumulation, investigate it and draw it accordingly.

Here Ana expressed an unexpected construct, namely that the rate is the derivative of the accumulation function and used it to draw the graph of the accumulation function.

## THE CASE OF ROY AND DON

When dealing with the constant flow rate of 30 litres per minute, Roy wrote: "The amount of water accumulated in the pool increases at the constant rate of 30 litres per hour, so we get a straight line of $y = 30t$" and verbally added "as Mike, likes to say" (Mike is Roy's physics teacher). Roy related the situation to his physics studies. He sketched the graph $y = 30t$ when asked to draw the accumulation function. It seems that Roy's conclusion, that the given flow rate is the slope of the accumulation function, is supported by his physics studies.

While working on the tasks related to the GeoGebra animation, Roy explained: "[the animation] gives a rectangular shape which has an area that represents the connection between the flow rate and the time of filling, whose product gives the added amount". He expressed that the area represents the amount, a construct he used in later tasks.

When drawing the accumulation function for the flow rate that is constant in segments, Roy calculated the added amounts as product of rate and time interval in each segment. When asked about the connection between the given rate graph and the accumulation function, Roy explained: "The graph of the flow rate represents the slope of the accumulation function graph at each $t$, that is, the graph of the derivative"; here he expressed the unexpected construct that the flow rate graph is the derivative of the accumulation function. He then elaborated: "it reminded me of kinematics… say someone has a velocity-time graph, so it is also segmented… say they tell you what the displacement is, so I build a graph accordingly. It's like algebraic expressions…".

When discussing the positive, linear, decreasing rate graph, Roy used the analogy from kinematics. When asked to draw the accumulation function, he concluded that it is a parabola because the given graph is the derivative of the accumulation function graph.

## DISCUSSION

Ana and Roy constructed the unexpected construct, namely that the given rate graph is the derivative of the accumulation function, while their partners apparently did not. In part 1, both Ana and Roy determined that the accumulation function is $y = 30t$. Ana reached this conclusion analytically, finding the algebraic representation of the function using two points. Roy reached it by analogy from contexts in his physics studies. When determining that the accumulation function is $y = 30t$, both Roy and Ana claimed that the given rate is the slope of the accumulation function. This observation might have transformed their thinking from describing the process qualitatively to reasoning quantitatively about the graphical representation.

Roy first expressed that the given rate is the derivative of the accumulation function in part 2, when reminded of a similar question in kinematics. He again expressed it in part 3, but did not make use of it to construct the accumulation function.

Ana first expressed the unexpected construct in part 3 and promptly made use of it to build the accumulation function. The construct may have emerged then for Ana because the strategy of multiplying the rate by the time duration did not work; looking for a different strategy to find the accumulation function, helped her connect slope (which she had noticed earlier) to derivative.

Both, Ana and Roy expressed in part 1 that the slope of the accumulation function was a straight line whose slope equals the given constant rate value. In part 2, this was reinforced. In part 3, they needed a different strategy. Ana achieved the strategy via the unexpected construct based on her previous knowledge about the derivative being the slope at every point. For Roy, it was anchored in other contexts from physics.

Throughout his work, Roy uses analogies from physics, which helped him to solve the tasks. The pool context may have supported meaningful reasoning and the application of ideas across contexts, an example of how real-world situations can foster context-dependent strategies and generalization (Gravemeijer & Doorman, 1999) and possibly help overcome known challenges in transferring mathematical knowledge (Jones, 2015). We cannot confidently attribute Roy's generalization solely to the effect of the pool context, since we know that he encountered similar problems in his physics class. However, Ana and Zoe, who did not study physics, also arrived at the same generalization. This may suggest that the pool context played a role in supporting such reasoning, even in the absence of prior exposure to analogous problems. In conclusion, in spite of some differences between them, Ana and Roy both made essential use of one (Ana) or several (Roy) EMCs to go beyond the intended knowledge construction process; in the first sequence of activities toward learning integration, they constructed the connection that the accumulation function of a given a rate of change function has the property that its derivative equals the given rate of change function - essentially the fundamental theorem of calculus.

## ACKNOWLEDGMENT

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# Basic Mental Models of the Integral: A Didactical Approach for Lecturers of Thermodynamics

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*Understanding calculus concepts, especially integrals, is critical in physical chemistry (PC) and narrower in thermodynamics, a field where these concepts pose significant challenges to students. This paper investigates the role of 'basic mental models' (BMMs) of the integral in supporting comprehension of core concepts in thermodynamics. Through an analysis of textbook content, we identified how the different BMMs are needed within thermodynamics. Results reveal that nearly each BMM of the integral is relevant for explaining certain thermodynamic concepts. These findings highlight the value of the mathematical-didactical concept of BMMs and emphasize its importance for PC lecturers in effectively conveying complex concepts and enhancing student understanding.*

*Keywords: physical chemistry, thermodynamics, integral, basic mental model.*

## INTRODUCTION

Physical Chemistry (short 'PC'), particularly within the central domain of thermodynamics, constitutes a fundamental element of undergraduate chemistry curricula. They are often regarded as one of the most challenging aspects of a typical chemistry degree program. They are based not only on physics but also on mathematics, especially calculus (David, 1995). A study conducted with Turkish students identified the primary challenges perceived in PC courses (related to the course itself), highlighting the 'abstract concepts', followed by a lack of 'deep understanding' and 'too much mathematics', among others (Sözbilir, 2004). Since calculus is a central part of thermodynamics, one might think that it is important to improve students' mathematical skills. And indeed, there is evidence that higher math skills are correlated with higher performance in PC in general (Hahn & Polik, 2004). Therefore, it makes sense to use the knowledge from mathematics didactics to support students in gaining a 'deeper understanding' in the main part of undergraduate PC lectures, namely thermodynamics.

In this article we demonstrate the use of the so-called *Grundvorstellungen* ('basic mental models', abbreviated as **BMM**), which have emerged from a decades-long discussion about what learners should understand about mathematical concepts (Greefrath et al., 2021). Here we focus on the mathematical concept of the integral, which plays a crucial role not only in calculus and thus in thermodynamics, but also in the STEM field in general. To this end, the subsequent discourse is structured according to the following research question: To what extent can concepts and terms of thermodynamics be interpreted with BMMs of the integral?

## THEORETICAL BACKGROUND

### Basic mental models of mathematical concepts

The BMMs of a mathematical concept describe the content-related interpretations that give meaning to that very concept (vom Hofe & Blum, 2016). A mathematical concept can have several different BMMs and together they "are prerequisites for dealing with [that] mathematical concept in an insightful way" (Greefrath et al., 2021, p. 650). The concept of BMMs allows both a *normative* and a *descriptive* work (vom Hofe & Blum, 2016). On the one hand, the normative BMMs act as 'learning objectives' and are determined by a didactic analysis of the mathematical concept. On the other hand, the descriptive work makes it possible to identify learners' individual BMMs which "are the specific manifestations of normative BMMs in a person" (Greefrath et al., 2021, p. 650). These can be derived from the learners' oral or written statements. They serve to indicate which of the individual BMMs need to be further developed in order to best correspond to the normative BMMs - and thus a comprehensive understanding of the mathematical concept.

### Basic mental models of the integral

Here we focus on the mathematical concept of the (definite) integral, which plays a key role in thermodynamics. Since the mediation and teaching of 'abstract concepts' and the acquisition of a 'deeper understanding' are the core issues, this article presents the normative BMMs of the integral (see table 1), which serve as an adequate interpretation of the mathematical concept (vom Hofe & Blum, 2016).

| **BMM of** | **Description** |
|---|---|
| *area* | The definite integral is interpreted as the oriented area enclosed by the function's graph and the $x$-axis. Areas below the $x$-axis are counted as negative. |
| *(re) construction* | The integral represents either the total variation of a quantity (given its rate of change) or serves to determine an antiderivative. "Construction" derives a new function from known values, while "reconstruction" determines existing relationships from rates of change. |
| *accumulation* | The integral is viewed as the limit of a sum of partial products, focusing on the accumulation process before reaching the limit. Geometrically, this corresponds to summing the areas of rectangles with infinitesimally small width. |
| *average* | The integral is interpreted as the average function value over an interval. The mean value theorem for Integrals ensures that there exists a point $\xi$ where this average value is attained. Geometrically, this corresponds to a rectangle with the same area as the integral. |

**Table 1: Normative BMMs of the integral as suggested by Greefrath et al. (2021).**

## METHODOLOGY

To answer the research question, all chapters of the (standard) textbook on PC by Atkins (2017) dealing with thermodynamics, namely Focus 1-5 and 13, were qualitatively analysed based on the structured qualitative content analysis according to Mayring (2022). Other chapters were not examined. All authors, coming from the complementary fields' didactics of mathematics and chemistry, analysed the chapters by looking for formulas and diagrams in which integrals appear or play an important role. These occurrences were then coded by using the normative BMMs of the integral (see table 1) as a coding scheme focussing on whether formulas or diagrams in these chapters can be interpreted using at least one BMM. Therefore, a single occurrence can be coded multiple times. All occurrences coded with the same BMM were then sorted in terms of thermodynamic concepts and terms.

## RESULTS

The results are presented separately for each BMM, in that sense that first an example is presented to explain the BMM coded and secondly the thermodynamic concepts and terms are named. The classical BMM of area is needed, for example, in the canonical context of the *Carnot cycle*. In the typical representation of the Carnot cycle in the pressure-volume (PV) diagram, two adiabats and two isotherms enclose a curvilinear bounded area, the area of which then gives the work done by the Carnot engine. This area can then be determined using integrals, since these are 'measuring' (oriented) areas. The same also applies to the representation in the temperature-entropy diagram, where the area is a rectangle, whose area indicates the amount of heat transferred. Altogether, the BMM of area is essential for all thermodynamic cycles and (some) other representations of work in PV diagrams (e.g. expansion work at constant pressure). When depicting these in PV diagrams or temperature-entropy diagrams, it is important to understand that the enclosed area denotes the work, which is represented by the (definite) integral.

The second example, for needing the BMM of (re)construction, can be found in the temperature dependence of the reaction enthalpy. If the reaction enthalpy is given at a temperature $T_1$, the reaction enthalpy at a second temperature $T_2$ can be determined (or rather 'constructed') using the integral of the heat capacity over the temperature differences (see '*Kichhoff's law*'). The integral can therefore be interpreted as the total variation of the reaction enthalpy over the interval of the temperature difference determined from the rate of change, which is given by the heat capacity. Generalized, the BMM of (re)construction is required whenever a state function is described by other state functions in the form of integrals (e.g. Gibbs energy with variation of pressure). The integrand may only depend on one state function that ultimately specifies the rate of change of the other state function, which in turn represents a total variation.

The BMM of accumulation is required when determining the *pressure-volume (PV) work* at non-constant pressure. Consider a gas in a piston doing PV work and the external pressure is not constant. For very small changes in volume, the pressure becomes almost constant. This is the essential core idea that leads to a product sum. Because when the work is accumulated, which is the product of pressure and volume, by all these small changes, the total PV work $(w)$ is determined. This can be calculated as $w = \int_{V_i}^{V_f} p\,dV$ where $V_i, V_f$ are the volumes at the initial and final points, respectively, and $p$ is the external pressure. This is also generalized in the functional relationships described for state functions as on the BMM of (re)reconstruction. This is because it can be interpreted as starting with only small changes and accumulating them. And making the changes smaller and smaller, the integrand becomes approximately constant.

Finally, the BMM of the average only (!) appears in the *van der Waals equation* for real gases. When isotherms of the van der Waals equation are plotted on a PV diagram, they deviate from the empirical values in a certain interval below the critical temperature; the so-called van der Waals loops. Therefore, these loops "are replaced by horizontal lines drawn so the loops define equal areas above and below the lines: this procedure is called the Maxwell construction" (Atkins, 2017, p. 24). The direction of thought here is therefore from balancing the areas to a suitable 'average function value', which can then be determined by using the integral. In contrast, the formulation of the BMM of average is from the integral to the average function value. Nevertheless, the focus is always on the average function value and its connection to the integral, where the former can be interpreted as the saturation vapour pressure in the van der Waals equation. No further examples for the BMM of average have been found.

## DISCUSSION

A first aspect of the research question can be answered to the effect that each of the four BMMs of the integral is needed in at least one thermodynamic example. A more comprehensive answer is that the general thermodynamic concepts of thermodynamic cycles, representation of work and the functional relationships between state functions can be interpreted with three BMMs of the integral (area, (re)construction and accumulation). There is no general concept which can be interpreted with the BMM of average, as this is the only BMM with only one occurrence. Since this is a textbook analysis, no data from learners can be analysed and thus only "normative work" can be done. However, this was a deliberate choice, as the content of the research question focusses on (normative given) thermodynamic concepts and terms and not on individual persons.

Finally, it should be noted that these results are exclusively of a subject-matter didactic nature and require evidence-based research. For this purpose, it is necessary to collect data from students of thermodynamics. This has recently been done with the test instrument of Greefrath et al. (2021) and will be analyzed in another article. A translation of the instrument can be found at Bhatt et al. (2025).

## CONCLUSION

For three of the four BMMs (area, (re)construction and accumulation), we were able to identify broader concepts and terms in the field of thermodynamics for which the BMMs of the integral promote with a better understanding. Our results highlight that the BMMs of the integral must not only be considered individually, but that different perspectives on the (same) integral must be emphasized in order to deal with integrals appropriately depending on the context in thermodynamic (cf. Greefrath et al., 2021). For example, in the context of describing state functions with another, on the one hand the 'motivation' for using the integral is given (BMM of accumulation), and on the other hand the functional relationships between the state functions are brought into focus (BMM of (re)construction). Therefore, various perspectives are necessary for learners to develop a deeper understanding. It is precisely for this reason that the mathematical-didactics concept of BMMs of the integral could help lecturers to adequately explain the thermodynamic concepts and terms described above. We would even go so far as to say that the concept of BMMs of the integral could be considered - even if only intuitively - as part of a PC lecturer's pedagogical content knowledge. The analysis shows that different approaches to an integral as described by the BMMs can be theoretical fruitful in the learning process.

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# Calculus quantities and conservation laws in physics

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*A focus on quantity in calculus instruction can enrich students' understanding of what they are doing, and why they are doing it. To help prepare calculus students for future learning in a physics course, we present an evidentiary argument that a Riemann sum representation of integration with quantities is important for quantifying in physics. This paper presents two conservation laws encountered in a physics course through the lens of the Fundamental Theorem of Calculus as a potential bridge between the calculus that students learn, and important physical contexts in which it is used.*

*Keywords: calculus, physics, fundamental theorem, quantity, infinitesimal.*

## INTRODUCTION

Physics is the science of change, and calculus is its language. Most physics and engineering majors are required to complete calculus and calculus-based introductory physics courses in their first year of study – ideally preparing them to use calculus in physically significant contexts. Calculus helps guide modelling in physics; it is essential to describing how physical quantities are related to each other, and for creating a structure for new ones to emerge. While many students master procedures in their calculus courses, research shows that it is not unusual for them to view the mathematics in mathematics courses as distinct from physics (Bajracharya, Sealey and Thompson, 2023). This paper argues for an agreed-upon objective for calculus learning that students understand *why* they do what they do in a calculus course, as well as how to do it. The physical world creates a *need* for the tools that calculus provides. This need is an opportunity for learning, as seen through the lens of Harel's (2008) necessity principle -- students must have an intellectual need for a topic to be able to learn it.

In addition to quantities playing an important role in physics, the quantities of calculus mean more in *calculus learning* than simply being the objects of procedures. Researchers argue that reasoning with an explicit focus on mathematical quantities facilitates students' learning of calculus. For example, the differential $dx$ in an indefinite integral is seen by many as a cue to the variable of integration. Operationally, there is nothing wrong with that interpretation, it helps you efficiently get an answer, but it reveals essentially nothing about why you would want to perform the integral in the first place. Many authors argue for an infinitesimal interpretation of $dx$ as a quantity, because it facilitates visualizing a tiny amount of something, which is valuable in making meaning of the ratio and products involving $dx$ (Thompson, 2011, Dray, 2016, Oehrtman and Simmons, 2023, Ely and Jones, 2023).

Ratio quantities, product quantities, rates, intervals, accumulation and change are mathematical quantities around which the ideas of calculus are formed. Student conceptualization of these quantities, and how derivatives and integrals emerge from their combination, is at the heart of understanding why one does calculus in the sciences, and not just how to do it. Conceptualizing the unit as part of quantity has been shown to be important for students in mathematics courses. Thompson (2011) emphasizes the importance of the unit as *part* of the quantity itself, e.g. a speed v= 10 m/s. In one study in determining the areas and volumes of shapes, Dorko and Speer (2015) observed that calculus students who wrote correct units could explain dimensions of planar figures and solids, and connect this knowledge to the shapes' units. In contrast, students who struggled with units also struggled with dimensionality.

This brief paper narrows the calculus focus to the evaluation theorem of the fundamental theorem of calculus (FTC) $F(a) - F(b) = \int_a^b f(x)dx$, and the mathematical quantities it combines. It highlights the generative richness of the FTC in the context of two foundational laws of physics – the laws of conservation of energy, and the conservation of momentum. The paper concludes, that through coordination, the two disciplines can help students' conceptual gap narrow.

## BACKROUND

### Quantification and symbolizing in physics

Quantification involves generating physical quantities as useful and productive objects for making sense of a situation. Consider a sailboat moving across water. What measurable quantities can help describe the motion? What arithmetic makes sense in constructing rates? What are their units? White Brahmia (2019) argues that quantification is at the root of modelling on physics, emphasizing the importance of quantity and its rate of change. Many physics quantities are vector quantities, and signed scalars, presenting an additional challenge for new learners. Many quantities emerge from multiplying and dividing other quantities (e.g., momentum = mass x velocity, density=mass/volume). Procedurally, the arithmetic involved in creating new quantities is not a challenge for most students, however understanding why the arithmetic makes sense can pose a significant challenge (Thompson, 2011).

Physics students struggle with symbolizing quantities and operations (Von Korff and Rebello, 2012). Given the challenges of quantification and symbolizing in introductory physics, students must have a solid understanding of mathematical quantities and their meanings before they blend them with the many new symbols they will encounter in a physics course. For example, Gauss's law combines mathematical quantities, symbolizing and both vector and scalar physical quantities: $\oiint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_o}$

### Reliable resources and difficulties applying calculus reasoning in physics

Conceptualizing the summing up of quantities, as exemplified in the Riemann sum, is productive for many students (Meredith and Margonelle, 2008, Von Korff and Rebello, 2012, Sealey, 2014, Ely and Jones, 2023). However, students often struggle with understanding physical quantities that approach zero. Visualizing what happens to the physical quantity represented by the infinitesimal *dx* in a definite integral in the limit that it goes to zero is difficult. Where does it go? What are you summing up if you're multiplying by zero? Research suggests that physics students are more successful when reasoning about summing finite infinitesimals, as this approach helps make the abstract concept of limits more accessible and intuitive. (Meredith and Margonelle, 2008, Von Korff and Rebello, 2012, Oehrtman and Simmons, 2023)

**Interpretation of the FTC through mathematics quantities**

Mathematics education researchers present a framing of the FTC as a relationship between key mathematical quantities of change, rate and accumulation (Thompson, 1994, Samuels, 2022), see Table 1(a).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Physics quantity** | $f(b) - f(a)$ | = | $\int_{x=a}^{x=b} df$ | = | $\int_a^b \frac{df}{dx} dx$ | = | $\int_a^b f'(x)dx$ |
| | Total change (accumulation) | | Infinite sum of small change | | Infinite sum of dep. variable change for each interval × interval | | Infinite sum of rate × interval |
| **impulse** | $p(t_2) - p(t_1)$ | = | $\int_{t=t_1}^{t=t_2} dp$ | | $\int_{t_1}^{t_2} \frac{dp}{dt} dt$ | | $\int_{t_1}^{t_2} F(t)dt$ |
| | Change in momentum | | Same as above | | Same as above | | Infinite sum of force × time interval |
| **work done *on* system** | $U(x_2) - U(x_1)$ | = | $\int_{x=x_1}^{x=x_2} dU$ | | $\int_{x_1}^{x_2} \frac{dU}{dx} dx$ | | $\int_{x_1}^{x_2} F(x)dx$ |
| | Change in potential energy | | Same as above | | Same as above | | Infinite sum of force × displacement |

**Table 1: (a) Shaded region represents mathematical quantities in the FTC (Samuels, 2022) (b) Unshaded region is an FTC representation of conservation laws.**

The term *change* here is used to refer uniquely to the change in the dependent variable. While both $\Delta y$ and $\Delta x$ are referred to as "change" in mathematics, in the context of scientific measurement they represent different types of change. One is manipulated, and the other is a response, even though they covary. The right –hand side is a sum of infinitesimally small products. Each product has a specific value of the rate as one factor, and infinitesimally small *interval* of the independent variable as the other. The key mathematical quantities are the *change,* the infinitesimal *products,* and the *accumulation* that is found through summing up these tiny products.

## EXAMPLES OF FTC IN PHYSICS: CONSERVATION LAWS

The conservation laws are introduced in students' first physics course, mechanics. The conservation of total mechanical energy and the total momentum of a system form the foundation of mechanics. The Fundamental Theorem of Calculus (FTC) provides a framework for representing these conservation laws, as represented in Table 1 (b).

The total energy of a system changes when an external object exerts a force on the system as its position changes along the direction of the force. The dot product of the two vector quantities (force and displacement) in the integral result in a scalar change in energy that is exactly equal to the cumulative effect of the force acting over a *displacement* (position interval). This accumulation is so significant that it is given a specific name: *work*. Work is the only way to change the mechanical energy a system. In the context in Table 1(b), the force both drives and quantifies the rate at which work is done as the object's position changes. Similarly, the total momentum of a system changes when a force acts over a *time interval*. The vector change in momentum is equal to the cumulative effect of the vector force over that total time interval. This accumulation, too, is so crucial that it is given a name: *impulse*. Here, the force both drives and quantifies the time rate at which momentum changes.
A student who is well-versed from calculus in the mathematical quantities that make up the FTC will be better-positioned to take up the new and challenging ideas that it frames when they encounter them in the context of physics. Energy and momentum are abstract, being able to fall back on a facility with conceptualizing the calculus can make learning them easier. Emphasizing the meaning of the operators, quantities and their symbols in the FTC can help prepare students to more easily frame the applications they will encounter in their subsequent courses.

| | **operators** | **quantities** | **language** | **physics examples** |
|---|---|---|---|---|
| change,<br>interval | $d$<br>$\Delta$ | $dy$<br>$\Delta x$ | dep. var. – change<br>indep. var. – interval | impulse as a change in momentum<br>displacement as a change of position |
| rate<br>of change | $\frac{\Delta}{\Delta x}$ $\frac{d}{dx}$ | $\frac{\Delta y}{\Delta x}$ $\frac{dy}{dx}$ | ratio<br>of change to interval | acceleration as the time rate of change of velocity<br>force as the time rate of change of momentum |
| accumulation | $\sum$<br>$\int_b^a$ | $\sum_i \left(\frac{dy}{dx}\right)_i dx_i$<br>$\int_a^b f(x)dx$ | sum of many small pieces | work<br>impulse |

**Table 2: FTC symbols and quantities common across calculus and physics.**

## IMPLICATIONS FOR CALCULUS AND PHYSICS INSTRUCTION

Symbols and quantities carry deep significance, and calculus instruction can convey that to students. Table 2 presents those that recur in the FTC, and merit instructional time in calculus. Physics instructors can help enrich their students' mathematical knowledge by knowing the calculus quantities and correctly using them in physics instruction help their students' calculus knowledge emerge in physics contexts.

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# Characterizing the Calculus Concepts and Skills Used in the Undergraduate Chemistry Curriculum: When Should Chemistry Majors Take Calculus?

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*In United States universities it is common for undergraduate chemistry majors to take calculus during their first two years; however, our work characterizing the calculus concepts and skills used in the undergraduate chemistry curriculum highlights that calculus is used sparingly until students' last two years when they take physical chemistry. The goal of this conference paper is two-fold: (1) present to the interdisciplinary audience our analysis describing the ways students are expected to use calculus across the undergraduate chemistry curriculum; (2) open a conversation regarding the alignment and relevance of chemistry students' coursework.*

*Keywords: chemistry; calculus; STEM; coursework.*

## BACKGROUND

Our interest in identifying the calculus used across the chemistry curriculum is based on the productive conversations that started at *The Learning and Teaching of Calculus Across Disciplines* conference in 2023. During this conference, Brian Faulkner gave a plenary that referenced prior work that evaluated the ways calculus was used in engineering (Faulkner et al., 2020). This work, as well as other work in an advanced mathematics context (Czocher et al., 2013), utilized the Calculus Content Framework (CCF), which was developed through consultation with experts to outline the specific concepts and skills students are expected to learn in a first-year calculus sequence (Sofronas et al., 2011). The utility of this framework is that it provides an operationalizable list of conceptual and procedural knowledge to serve as a lens to view the chemistry curriculum.

Within the United States, the university chemistry curriculum typically involves students starting with general (introductory) chemistry, followed by four course sequences that reflect the historical division of chemistry into subfields based on the topic of inquiry: organic chemistry, inorganic chemistry, biochemistry, analytical chemistry, and physical chemistry. Mirroring the expert-consensus framework in calculus, in this work we used the Anchoring Concepts Content Map (ACCM) developed by the American Chemical Society (Murphy et al., 2012). Created through multiple iterations with feedback from experts and comparison of alignment with exams and textbooks, the ACCM provides a fine-grained overview of the target knowledge students should learn in the chemistry curriculum. There is an ACCM for each of the chemistry sequences listed above (except for biochemistry, where we used a working draft) and they are organized using a tiered system that starts with larger concepts that unify all the course sequences at Level 1 and 2 and becomes increasingly more specific for each subdiscipline at Level 3 and 4 (Figure 1). The ACCM provided a useful approach to systematically move through the chemistry curriculum and consider the ways constructs from the CCF were used in relation to each target chemistry idea.

| Discussed in Undergraduate Chemistry Curriculum | | Discussed in Specific Course Sequences | |
|---|---|---|---|
| **Level 1**<br>Big Idea | **Level 2**<br>Enduring Understandings | **Level 3**<br>Subdisciplinary Articulations | **Level 4**<br>Content Details |

**Figure 1: Anchoring Concepts Content Map (ACCM) structure, adapted from Murphy et al. (2012).**

## CALCULUS CONCEPTS & SKILLS IN CHEMISTRY

Using the ACCM for chemistry and the CCF for calculus, our analysis looked at the general alignment of the frameworks by systematically working through the ACCM for each course sequence and coding instances where any of the calculus concepts or skills were used in relation to a chemistry Level 3 concept. As part of this, when it supported the analysis, we referenced the finer-grained Level 4 to inform our characterization. To build a case for the trustworthiness of our qualitative analysis (Watts & Finkenstaedt-Quinn, 2021), we began by comparing our characterization of calculus use with standardized exams developed by the American Chemical Society Exams Institute, which has exams for each course sequence and are commonly used as formative assessments in the United States, reflecting the expected knowledge and skills students should have after completing a course sequence. Following this, we met with a different faculty expert currently teaching each course (e.g., general chemistry, organic chemistry, inorganic chemistry, biochemistry, analytical chemistry, physical chemistry) and discussed our analysis, asking about their experiences regarding the expected calculus use and how that compared with our tentative findings. As demonstrated in the findings below, due to the general lack of calculus used in most of the chemistry courses, this process worked well for validating our analysis –except for physical chemistry. The large amount of calculus concepts and skills in physical chemistry necessitated additional work in which following our initial coding we had a chemistry education research colleague with physical chemistry expertise independently code the physical chemistry ACCM. Each calculus concept and skill code assignments were compared, and discrepancies were discussed until consensus was reached.

### CALCULUS IN PHYSICAL CHEMISTRY

Physical chemistry can be thought of as an important capstone to the undergraduate chemistry curriculum, focusing on the complex mathematical models that explain chemical phenomena; however, until students take this course, our work indicates they are primarily using pre-calculus across the other course sequences in chemistry. One of the important disclaimers of our work is that we do not intend to suggest that students are not using mathematics in their chemistry courses, but rather that the use of calculus is not common until they take physical chemistry. Due to space constraints, we will only briefly comment on the use of calculus in the entire chemistry curriculum, focusing our discussion on calculus use in physical chemistry.

Across all five ACCMs, there were a total of N=164 code assignments, with each code reflecting a topic in which students are expected to use calculus concepts or skills in the chemistry curriculum. Of this code total, 93% of the codes (N=152) were applied to the physical chemistry ACCM. The few instances where calculus was discussed often involved students using calculus concepts (e.g., reasoning with and about the derivative, as opposed to differentiation calculations) to draw conclusions about big ideas such as chemical kinetics (an area of study emphasizing the rate of chemical reactions). Given the large weighting toward calculus use in physical chemistry, the remainder of this discussion will focus on trends noted in relation to the calculus concepts and skills used within this course sequence.

Focusing on the codes applied to the physical chemistry ACCM (N=152), most of the expected use of calculus was related to concepts (81%, n=123) instead of skills, with the specific distribution of concepts and skills provided in Figure 2. Notably, the application of derivatives and integrals were the most common concepts and skills. As a disclaimer, within the Calculus Content Framework fundamental theorem limit, Riemann sums, and continuity – calculus concepts – and epsilon-delta, definitions, limit calculations, and parametric equations – calculus skills – were not identified in physical chemistry (or any course across the chemistry curriculum).

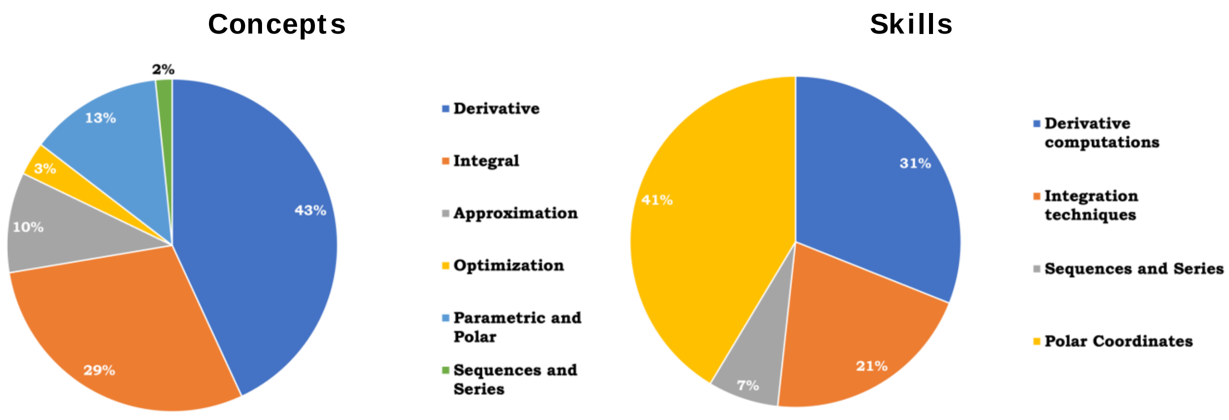


**Figure 2: Calculus concepts (n=123) and skills (n=29) from the Calculus Concepts Framework identified in the physical chemistry Anchoring Concepts Concept Map.**

In terms of the common contexts where students were expected to use calculus, Although chemical kinetics, was the primary context where calculus was used across the other courses sequences, within physical chemistry, the first three big ideas (that emphasize the quantum mechanical treatment of atoms, bonding, and structure/function) served as the context for 48% (n = 74) of the calculus codes assigned.

## DISCUSSION AND IMPLICATIONS

Introductory STEM courses are often referred to as *gateway* courses because of the way they serve as an entrance into STEM; however, many comment on the resulting *gatekeeping* effects of these courses that prevent students from remaining in STEM, suggesting a potentially high-yield context for intervention (Hatfield et al., 2022; Matz et al., 2018). In the context of chemistry, perhaps unsurprisingly, there is a large body of research that correlates student success with mathematical ability (Ralph & Lewis, 2018), including a long history of using standardized college-entrance mathematics exams to predict students' success in general chemistry (Spencer, 1996). Therefore, for students to be successful in college chemistry, they need ample mathematics preparation, which is not guaranteed at the secondary level. That said, it is worth reflecting on whether current degree programs provide enough time for students to develop the mathematical concepts and skills necessary for success.

For example, consider the typical four-year plan for chemistry majors at the University of Wisconsin – Milwaukee (UWM) located in the midwestern United States. For students to take physical chemistry as a capstone chemistry course during their third and fourth year, students need introductory physics as a perquisite, but since physics is calculus-based, students first need to take calculus. As an implication, chemistry majors are encouraged to take two semesters of calculus during their first year, with two semesters of introductory physics during their second year. This timeline does not provide much flexibility, suggesting the need for students to be calculus-ready when entering college; chemistry majors that first take college algebra and precalculus begin their college career "behind". UWM is part of a larger 13-university state system; each university has similar mathematics and science requirements for their chemistry majors.

Based on our characterization of the calculus concepts and skills students use in the chemistry curriculum, we aim to continue the conversations had at the previous CalcConf, promoting a much-needed interdisciplinary discussion regarding when chemistry students should take calculus. This conversation involves valuable input from both experts across fields as we consider the possibility of modifying the traditional course sequence and how that impacts coursework across disciplines. As part of this, we suggest modifying the typical four-year plan to delay taking calculus and communicating the importance of starting with college algebra and pre-calculus even if students "tested out" of these courses; however, this would only work if students take algebra-based physics instead of calculus-based physics.

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# Chemistry contexts for introducing derivative and integral

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*Mathematical notions, including the fundamental calculus concepts of derivative and integral, are often taught abstractly. For science students, such abstract presentation is likely to increase difficulties. The question thus arises whether teaching these notions in extra-mathematical contexts leads to more profound and more domain appropriate conceptualizations. But even before research studies can approach these empirical issues, the question arises how to choose appropriate contexts. This paper aims to contribute to the discussion about the choice of contexts; it uses chemistry as content domain, but analogous questions arise in physics, biology, engineering and economy.*

*Keywords: derivative, integral, conceptualization, chemistry contexts.*

## INTRODUCTION AND BACKGROUND

One of the lessons of CalcConf2 was that pure mathematics can be abstract, presenting conceptualization difficulties for science students, whose grasp of calculus concepts thus often remains superficial. Literature reviews suggest interdisciplinary research to better support students in generating constructive meanings for these abstract, concepts (Towns et al., 2025). This is coherent with recent literature: Rabin et al. (2021) wrote that "The 'practical' mathematics used by scientists differs in significant ways from the 'abstract' or 'structural' mathematics taught by mathematicians" (p. 9). Rubel and McCloskey (2021) listed rationales to contextualize mathematics; these include affective benefits such as motivating learners and helping them see the practical value of mathematics, but also the potential to serve as conceptual anchors (Reinke et al., 2020), through which students make sense of new mathematical concepts.

The use of extra-mathematical contexts (EMCs) has been proposed for introducing high school students and science majors to the fundamental concepts of calculus. For example, Elias et al. (2025) report that experts (mathematics educators and experienced high school teachers) have identified four characteristics of EMC based situations suitable for introducing the concept of derivative as rate of change: time is the independent variable, and the situation is familiar, tangible (i.e., concerns a concrete process that one can see or touch), and mathematically neat (i.e., only the independent variable influences the dependent variable).

We take as working assumption that it is desirable to use science contexts to either introduce from scratch or enhance (henceforth, we will abbreviate these two options as "introduce") science students' conceptualization of fundamental calculus concepts such as the derivative and the integral. We consider treatments of these two notions in a way that is mathematically coherent as well as constructive for use in the students' scientific content domain. We focus on chemistry.

Kent and Stevenson (1999) have shown that chemistry students' conceptualization of the integral may be sub-optimal, and hence not as constructive as desired. They investigated how undergraduate chemistry students in a laboratory situation were able to connect their science knowledge that "integrating the force gives the potential energy" to their notion of integral. These were students who had learned about integrals in their mathematics courses and were asked about their use of integrals in chemistry contexts. The findings of Kent and Stevenson (1999) were that the connection was tenuous; while students did make a qualitative visual connection between energy and area under a graph, this connection had little analytic content. The integral was a computational tool in the current situation, and students were not able to make use of the underlying ideas in other situations.

## AIM

This paper presents a didactical reflection by mathematics educators on the use of situations from chemistry for introducing students to two fundamental notions of calculus, the derivative and the integral. Towns et al. (2025) surveyed situations where calculus is used in the undergraduate chemistry curriculum. Among these, they chose rate laws for developing a treatment of the derivative as rate of change in a chemical context; they chose the expansion of an ideal gas for developing a treatment of the integral as accumulation in a chemical context. In the next two sections, these two contexts will be examined in view of the four characteristics proposed by Elias et al. (2025); the treatments will also be analysed as to their structural analogy to the mathematical concepts. As such, the paper does not present an empirical research study but is intended to serve as basis for a more refined discussion than was feasible at the previous conference.

## A CHEMISTRY CONTEXT FOR THE DERIVATIVE AS RATE OF CHANGE

Rate laws describe the concentration of a substance during a chemical reaction as a function of time. Chemistry students typically learn about rate laws by measuring the concentration of a reactant at a sequence of points in time and graphing the measurements. They are also told that the rate is usually of the form $kC^n$ (different notations are used in chemistry), where $C$ is the concentration, the typical values of $n$ are 0, 1, or 2 for so-called zero order, first order or second order reactions, and $k$ is a (negative) constant whose dimension depends on $n$.

Towns et al. (2025) propose to introduce students to difference quotients $\frac{\Delta C}{\Delta t}$ (without using the $\Delta$ notation), starting from the case $n = 0$, and continuing with $n = 1$, explaining why the difference quotient is a measure for how quickly the concentration changes, explicitly calling it the rate of change of the concentration in the relevant time interval, and asking students to compare rates of change in different time intervals. They then note that while the empirical rates are based on a discrete (and not necessarily regular) sequence of points in time, the chemical process is continuous and one might therefore be interested in the (instantaneous) rate of change at any specific point in time, and in how this instantaneous rate of change can be found; this leads to mathematicians calling it the derivative of the concentration at that point in time, and to finding the rate constant $k$ using the derivative (see Towns et al. 2025 for details).

The above is an attempt to introduce chemistry students to the notion of derivative in a context that is familiar to them from the laboratory as well as tangible: They have carried out the measurements. While for some chemicals concentration may not be very tangible (and for others, say an acid, one might not want to touch), chemists could think about using a substance with a strong colour so that students could not only measure but actually *see* the concentration. In the chosen context, the independent variable is time: The concentration varies with respect to time. And the influence of external variables is negligible; of course, temperature, for example, may have an influence on chemical reaction rates, but chemistry teachers are interested in setting up the laboratory for beginning students so that such effects are of no influence on the process. Hence, all four criteria of Elias et al. (2025) are rather well satisfied. This context teaches us that the four criteria may be intertwined. Familiarity, tangibility and external variables all seem to depend on how the laboratory situation is set up.

Finally, to motivate the notion of derivative, it is desirable to use a context that offers a structural analogy with the mathematical process of transition from the algebraic notion of difference quotient to the calculus notion of derivative. This transition should make sense or even be desirable from the point of view of the context. Here it is the continuity of the chemical process (as opposed to the discreteness of the points in time at which the concentration has been measured) that motivates making the time intervals shorter, in fact as short as we please. An analogy thus exists between the chemical context and the mathematical transition to the limit. Students may experience how the mathematical notion of derivative models the very short but not sharply defined time intervals, about which chemists might think.

## A CHEMISTRY CONTEXT FOR THE INTEGRAL AS ACCUMULATION

Chemists describe $PV$-work as the work done on (or by) a frictionless movable piston as the volume $V$ of a fixed amount of gas changes at a constant temperature $T$ against an external pressure $P_e$ on the piston. Chemistry students are typically asked to examine the work done during such an expansion in several steps, each taking place against an external pressure $P_e$ and doing work $P_e \cdot \Delta V$, where $\Delta V$ is the change in volume during the step. Here $P_e$ is taken to be constant during each step, and somewhat smaller than the gas pressure $P$, which is why the gas is expanding. Figure 1 shows a graphical representation of the work done in a three-step expansion. The grey rectangle areas represent the work that accumulates as the gas is expanding. Students compare the work done in one-step, two-step and three-step expansions, and to discuss how to maximise the work done. This naturally leads to increasing the number of steps, while decreasing the difference between the external pressure and the gas pressure at each step. The limit case obtained in this way is important

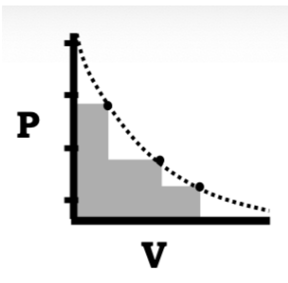


**Figure 1: Three-step expansion**

for the chemist: It is reversible since the two pressures are equal for every value of the volume. The increase in the number of steps is structurally equivalent to the passage to the limit in a Riemann sum; this provides a structural analogy between the chemical situation and the mathematical concept. To make the mathematics more concrete and the computations feasible, the behaviour of the gas may be approximated by the ideal gas law $PV = nRT$, that gives the gas pressure $P$, when $T$ is the temperature, $n$ is a measure for the amount of gas, and $R$ is a constant. We refer the reader to Towns at al. (2025) for a more detailed description of the introduction to integration in the chemistry context of $PV$-work.

With respect to the four characteristics in Elias et al. (2025), the ideal gas expansion context has volume rather than time as independent variable. The students' experience with the context is likely to contribute to their familiarity with it; on the other hand, the physics notion of work is quite abstract and moreover the situation requires the transition from work as force along a path to the work $P_e \cdot \Delta V$ done by the gas on the piston. The context is  tangible – in a suitable experimental situation one can clearly feel the pressure exerted by the gas. External variables may be a disturbing factor; on the one hand, the temperature $T$ and the amount $n$ can be kept quite constant; but on the other hand, the gas is real and might deviate from ideal gas law.

## CONCLUDING REMARKS

We considered two contexts from chemistry as EMCs to introduce abstract mathematical concepts, the context of a chemical rate law for an introduction to differentiation and the context of the expansion of a gas for an introduction to integration. In each case, we have pointed to the structural analogy between the chemical process and the mathematical concept. We suggest that such a structural analogy is a crucial prerequisite for the successful use of the EMC for introducing a mathematical concept, and that such an analogy may support students in using the EMC as conceptual anchor in the sense of Reinke (2020).

We also analysed the two contexts in terms of the characteristics of EMCs proposed by Elias et al. (2025). We found that rate law context satisfies these characteristics to a much larger extent than the ideal gas context. A main such characteristic is time as the independent variable. People naturally tend to think of processes in daily life or in science such as the expansion of a gas or a chemical reaction in terms a change happening in time. Rates of change with respect to other variables than time, while mathematically equivalent, do not correspond to our experience. It is an empirical question, what influence this has in practice.

We have used the domain of chemistry. The question arises whether the situation is substantially different in other content domains. While chemistry offers some suitable contexts for calculus concepts, the choice is limited to a few subdomains of chemistry (Towns et al., 2025); physics and engineering might have a larger number of suitable contexts, while biology and economic might have fewer. Moreover, while issues of discretization are present in all sciences, they might be harder to overcome in economics, because in many contexts of the natural sciences and engineering, the underlying processes are continuous as are the ones in this paper, and the discretization occurs because of the discrete nature of laboratory measurements, but in economics, the underlying processes are often discrete themselves.

Finally, a didactic issue when using EMCs concerns the desirable number of contexts. The use of a single EMC might tightly link the mathematical concepts introduced to this specific EMC. The use of more than one EMCs might enhance the potential productivity of the students' meanings, but it might, on the other hand, impose a high cognitive load. Will the gain from linking between a variety of EMCs be lost by the increased cognitive demands on the students? These are issues for empirical research raised by the attempt to use EMCs to introduce basic abstract mathematical notions.


## ACKNOWLEDGEMENT

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## NOTE

1. Towns et al. (2025) has been submitted to the CalcConf2 Special Issue of the *International Journal of Research in Undergraduate Mathematics Education*. It is available from the authors.

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# Connecting Mathematics and Medicine in a Calculus Course in Biomedical Engineering Degrees through SRPs

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*Abstract. Calculus courses in Biomedical Engineering degrees in Spain and Portugal do not usually relate the mathematics taught in the classroom with possible applications in the medical field. To overcome this lack of connection, we present two proposals based on inquiry-based learning, the so-called Study and Research Paths, which have been implemented in the last two academic years at a Spanish university. Both proposals consider the concentration of a given drug in the bloodstream of a certain individual, in which the relationship between functional modelling and elementary differential calculus lead to answer to the medical problems addressed.*

*Keywords: Anthropological Theory of the Didactic, Biomedical Engineering degrees, Modelling, Calculus, Study and Research Path.*

## INTRODUCTION

In recent years, the number of health-related technical degrees has increased significantly. According to WHO (2017), 67% of Biomedical Engineering degrees (BMED) in Europe have been introduced since 2000. This increase is mainly due to the need to train professionals capable of facing the current challenges resulting from the continuous technological advances that impact medicine. These studies must consolidate knowledge in both the engineering and medical fields, so a number of subjects requiring a high level of mathematical knowledge are compulsory. Hence the inclusion of Calculus course in BMED.

In the particular case of BDEM in Spain and Portugal, the syllabi include, as a minimum, the study of real functions of a real variable (domains, continuity, derivability and integrability) and present a rather standard organisation. There is hardly any reference to biomedical applications, which evidences an attitude of 'applicationism' as pointed out by Barquero (2013). Moreover, lecturers usually do not have a background in health sciences, which makes it difficult for them to propose applications outside the syllabus. In addition, in none of the syllabi consulted is there any reference to a book on mathematics for biomedicine.

However, although the contents description and bibliography do not refer to possible links with medicine, the 'learning outcomes' always allude to such applications. In fact, there are numerous references to future applications, to be carried out once both the contents and solution techniques have been assimilated – provided there is time to do so –, but never as a starting point for biomedical engineering problems. Precisely this contradiction leads to our research question: How can the content of a Calculus course be articulated with medicine in BMED in a more functional way?

There is not much research on teaching and learning processes in BMED, as in other engineering degrees (González et al., 2022; Faulkner et al., 2020; Pepin et al., 2021). These studies agree on the need to incorporate mathematical modelling into these processes, as it allows the mathematics taught in the classroom to be related to its future application in a specific field. On the basis of these studies, we present two different experiences carried out over the last two years in a BMED at a Spanish university.

## METHODOLOGY

In order to incorporate mathematical modelling and put it at the centre of the study process, we use inquiry-based learning proposals based on the tools offered by the Anthropological Theory of the Didactic (ATD). In particular, we use didactic devices called Study and Research Paths (SRPs) that have been implemented for two decades in several universities (Chevallard, 2015; Bosch, 2018). SRPs have shown to play an effective role in articulating the mathematical contents of a first university course and connecting them with future applications (Barquero et al., 2018; Barquero et al., 2021; Florensa et al., 2018; Florensa et al., 2019; Fonseca et al., 2014).

The SRPs are based on a generative question proposed to the students, which is related to their professional future and has no direct answer. The main goal is that students work out a final answer to the initial question through a sequence of derived questions and their associated answers. The mathematical tools that enable them to progress along their path are not always known, and their introduction are therefore justified. Students work in small groups and through a process of inquiry take on different responsibilities and combine moments of studying of the available information with moments of research (Winslow et al., 2013).

## EXPERIMENTATION AND RESULTS

We present two SRPs that have been carried out with first-year BMED students in a Calculus course at a Spanish university in the academic years 2023/24 and 2024/25 (Serrano, 2024; Serrano et al., 2024). In this degree all courses are characterized by a case study methodology, conditions that facilitated the implementation of the SRPs. The SRPs run in parallel to - but separately from - the traditional theory and practical classes.

These SRPs are partially derived from a reference epistemological model based on the work of Lucas (2015), built around the relationship between functional modelling and elementary differential calculus in the transition between secondary education and university, focusing in particular on the Spanish and Portuguese education systems (Gascón, 2014; Lucas et al., 2019; Florensa et al., 2020).

In both cases, the SRP was related to the concentration of a particular drug in the bloodstream. In one case, the initial question guiding the research process was how to control the anticipation of a certain dose of a regularly administered drug. In the other case, the question concerned the change in the amount of nicotine when smoking a cigarette. The students had to find functional models characterising the systems (exponential functions, surge functions, etc.) that depend on different parameters. Precisely, SRPs promote work with parameters, which is a type of activity that is not usually within the domain of students.

The analysis of the behaviour of these functional models led them to establish relationships between variations or behaviour – medicine – with derivatives or limits – mathematics – and to study how different parameters influence the characteristics of their models, and are related with physiological characteristics. In both cases, new questions must be posed to answer the initial question, whose solutions open the way to the final answer. The use of mathematical software such as GeoGebra was useful in the activity.

In addition to the class work, students had to periodically present their results in a written report as part of the course evaluation, and they had to fill in different questionnaires that allowed lecturers and researchers to evaluate the work done during the SRP.

## CONCLUSIONS

The syllabi of the Calculus course in Biomedical Engineering degrees in Spain and Portugal are very similar to those offered in any other engineering degree and hardly present applications to the medical field. Moreover, the lack of lecturer training in the medical field, together with the scarcity of available bibliography, makes it difficult to offer these applications. This situation makes it challenging for students to understand the usefulness of mathematics in their professional future.

To overcome these issues, we present two SRPs implemented in the last two years, which have proven to be a relevant tool to answer our research question. Both SRPs facilitate the relationship between functional modelling and elementary differential calculus from a biomedical problem, while establishing connections between mathematics and medicine.

The students have valued the activity very positively, as they have managed not only to understand the usefulness of mathematics in their future professional practice, but also to carry out a type of mathematical work that is very different from what they are used to, adopting roles that are normally attributed to the lecturer, such as deciding paths and strategies to follow, working with parameters or presenting results in the form of a report.

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# Differentials across disciplines: Conceptions used in economics, physics, and chemistry

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*Although mainly supplanted by the limit definition in modern mathematics, differentials are commonly applied in other STEM disciplines and in economics. In this exploratory study, we investigated how differentials are understood in different disciplines: macroeconomics, chemistry and physics. Our results point out several conceptions of differentials that are used across these disciplines, while others seem more discipline specific.*

*Keywords: differentials, ways of understanding, teaching and learning of specific topics in calculus, mathematics practices related to calculus across disciplines.*

## INTRODUCTION

Differentials played an important role in the early development of calculus, but were later largely abandoned in favor of the limit approach considered more logically sound (Ely, 2021). Yet, they are still commonly applied in economics and the STEM disciplines, such as physics, where they are not only used for historical reasons but also because they can make some calculations simpler. More importantly, these disciplines come with their own contextualized interpretations, for instance, as an infinitesimally small increment in physics (López-Gay et al., 2015) or the change due to an increase by one more unit in economics (Feudel & Skill, 2024). These interpretations do not fit to the way differentials are usually introduced in calculus. Such discrepancies can lead to gaps for students and could have a negative impact on their understanding of the mathematical notions. Therefore, it is important to explore how mathematical concepts are understood and used in other disciplines in order to bridge such gaps.

In this article, we want to present an explorative study in which we investigated how differentials are understood in different disciplines, namely economics, chemistry, and physics. This is important in order to identify the meaning of differentials that students of these disciplines are expected to draw upon in their major subjects. Despite its importance, this issue has not been investigated thus far, as the focus of previous research on differentials lay mostly on students' conceptualizations.

## THEORETICAL FRAMEWORK AND METHODS OF THE STUDY

In the study presented here, we intend on extending the work on differentials done by Feudel and Skill (2024), who investigated how differentials are understood in microeconomics. They grounded their study in Harel (2008)'s framework using the construct *way of understanding*, which he defined as follows:

A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. (Harel, 2008, p. 490)

As Feudel and Skill (2024), assuming "that thinking cannot be separated from communication", we consider textbooks as "signifiers of the cognitive products of [the] mental acts [of the authors]" (pp. 5–6). In other words, explanations provided in the textbooks indicate ways of understanding differentials the authors draw upon.

Based on an analysis of microeconomics textbooks, Feudel and Skill (2024) identified nine ways of understanding differentials relied upon in microeconomics labeled as conceptions in Figure 1.

| Conception | Description |
|---|---|
| 1. Differentials as a mere sign for derivative or slope | The differentials appear only as a quotient $\frac{dy}{dx}$, which is used as a notation for the derivative or slope *without a contextual interpretation in the whole unit of analysis*. |
| 2. Differentials as an operator | The differentials appear in the form $\frac{d}{dx}(\dots)$ that requests to differentiate a specific expression. |
| 3. Differentials as shorthand for limits | The differential quotient $\frac{dy}{dx}$ is used as a notation for the limit of the difference quotient. |
| 4. Quotient of differentials interpreted as a rate of change | The differentials appear only as a quotient $\frac{dy}{dx}$, and this quotient is interpreted as a rate of change—verbalized as rate or change *per* unit. |
| 5. Quotient of differentials with discrete interpretation | The differentials appear only as a quotient $\frac{dy}{dx}$, and this quotient is interpreted as the change in $y$ when increasing $x$ by *one unit.* |
| 6. Differentials as symbols to be manipulated | The differentials appear as stand-alone objects and are treated as symbols that can be manipulated according to certain rules (e.g., shifted in equations) without contextual interpretation in the unit of analysis. |
| 7. Differentials as variables in a linear approximation | The differentials appear as variables in the linear approximation $dy = f'(x)dx$ with $dx = \Delta x$, or the author relates to a linear approximation in connection with differentials. |
| 8. Differentials as infinitesimally small quantities | The differentials are interpreted as infinitesimally small quantities (the term 'infinitesimal' is mentioned). |
| 9. Differential as a finite increment | A differential is interpreted as a finite change – maybe labelled as small, tiny, or marginal (without reference to linearity or the term 'infinitesimal'). |

**Figure 1: Ways of understanding differentials relied upon in microeconomics textbooks (Feudel & Skill, 2024, p. 9)**

In this explorative study, we extend the focus on: (1) Macreoeconomics — the other core area in economics study programs besides Microeconomics; (2) Chemistry; (3) Physics. In order to find out how differentials are understood in these fields, we used the conceptions from Figure 1 as "reference conceptions".

Similar to Feudel and Skill (2024), we also analyzed textbooks: one for each field. For chemistry and physics, we chose two textbooks that are widely used according to teachers of the disciplines at the authors' institutions (Atkins et al., 2023; Serway & Jewett, 2019). Concerning macroeconomics, however, the most common textbooks evade calculus. Therefore, we chose for this area a book that explicitly used calculus (Qian, 2023).

## RESULTS OF THE STUDY

### Conceptions of differentials used in the analyzed macroeconomics textbook

In the analyzed macroeconomic textbook (Qian, 2023), the occurrences of differentials are rather limited. Nevertheless, several conceptions from Figure 1 are used.

First, quotients of differentials occur several times in the textbook. These are mainly just used as symbols for the derivative (conception 1 in Figure 1). Only once is a differential quotient interpreted in the context in the definition of the marginal propensity to consume ($MPC = \frac{dC(Y)}{dY}$) as "the amount of additional consumption given unit increase in disposable income" (p. 53). Here, the differential quotient is interpreted as the change in $C$ when increasing $Y$ by one unit (conception 5 in Figure 1).

Sometimes, differentials also occur as single objects. In this case, they are considered as finite increments and as symbols that can be manipulated according to certain rules, for instance in the chapter on the IS-LM-model describing relationships in the market for goods and the money market. Here, Qian (2023) uses differentials in a symbolic derivation of certain equations (pp. 129–132). The IS-curve, for example, describes in the equilibrium of a marked for goods (i.e., demand equals income) the dependence of the gross income $Y$ from transfers $T$ like taxes, the government expenditure $G$, and investments $I$. It is given by the equation $Y = C(Y - T) + I(Y, r) + G$ with $C$ being the consumption and $r$ the interest rate. The author then argues that total differentiation yields $(1 - C'(Y - T))dY - I'(r)dr = dG - C'(Y - T)dT$, explaining that one may regard $dY$ and $dr$ as changes in $Y$ and $r$. Hence, the author uses certain rules to derive the final equation and interprets the differentials as finite increments (conceptions 6 and 9 in Figure 1). He also later uses them as variables in a linear matrix equation.

### Conceptions of differentials used in the analyzed chemistry textbook

In the analyzed physical chemistry textbook (Atkins et al., 2023), the conception of differential as an infinitesimally small quantity is very prominent. For example, when treating the first law of thermodynamics (p. 37 ff.), Atkins first explains that changes in the total energy $U$ of a system that is closed, i.e., that cannot transfer matter with its surroundings, only result from work $w$ done on the system and heat $q$ transferred to it: $\Delta U = q + w$. He then introduces differentials in this context as follows (p. 37):

> To take the discussion further and open up to the full power of thermodynamics, it is necessary to rewrite that equation in terms of an infinitesimal change in the internal energy $dU$. Thus, if the work done on a system is dw and the energy supplied as heat is $dq$, then $dU = dq + dw$.

Hence, the differentials are understood as infinitesimally small quantities (Figure 1, conception 8). Sometimes, however, Atkins also communicates about differentials as if these represented finite changes (Figure 1, conception 9), for instance in the context of entropy changes of a system that occur as a result of physical or chemical changes, such as vaporization.

A specificity regarding the usage of differentials in chemistry that we did neither find in physics nor in economics is the distinction between exact and inexact differentials. Atkins first explains that there is an important difference between *state functions* like the internal energy and *path functions* like heat or work (p. 58). Changes in a system's state can occur in multiple ways ("along different paths"). For instance, a system can be transferred from one state to another (e.g., a change in the temperature) by performing work or supplying heat. The change in the internal energy, however, does not depend on the path, but only on the initial and final states of the system. In this case, the differential of the quantity is called *exact differential*. However, the changes of $q$ and $w$ occurring in the system are different for these two paths. In this case, the differentials are called *inexact,* which Atkins defines as follows (p. 59):

> An inexact differential is an infinitesimal quantity that, when integrated, gives a result that depends on the path between the initial and final states.

Afterwards, he explains that inexact differentials are often even written with other symbols, for example $đq$ instead of $dq$.

**Conceptions of differentials used in the analyzed physics textbook**

Most of the conceptions from Figure 1 are present in the analyzed physics textbook (Serway & Jewett, 2019). The two exceptions are the differentials as a linear approximation (conception 7) and the discrete interpretation of the differential quotient as the change in $y$ when increasing $x$ by one unit (conception 5). The most frequent conceptions from Figure 1 consist in a symbolic use of the differentials: in $\frac{dy}{dx}$ as a notation for the derivative or slope only (conception 1), in the form $\frac{d}{dx}$ to differentiate some expression (conception 2), or as symbols to be manipulated (conception 6).

Other conceptions emerged in physics that were not present in the other analyzed textbooks, that of the differential as a small amount quantity and as a discrete punctual quantity. They distinguish themselves (1) from the conception of the differential as an infinitesimal quantity (conception 8), as they are not expressed as infinitesimally small, and (2) from the conception of the differential as a finite change (conception 9), as they do not convey the idea of change. For example, in computing the electric potential due to a uniformly charged disk, Serway and Jewett (2019, p. 649) explain:

> The calculation of the electric potential at any point $P$ on the $x$ axis is simplified by dividing the disk into many rings of radius $r$ and width $dr$, with area $2\pi r\ dr$.

Here, the differential $dA = 2\pi r\ dr$ (see Figure 2a) does not represent a small change in the area, but a potentially small element of area that would add up to form the whole disk. The conception of the differential $dA$, represented as a shaded ring in Figure 2a, is not the same as the conception of the differential $dm$ represented in blue in Figure 2b which illustrates the "parallel-axis theorem" – a result useful in the calculations of moments of inertia. Here, the authors specify that "the mass element $dm$ has coordinates $(x', y', 0)$" (p. 265), which reinforces the idea of the differential representing a discrete or punctual quantity.

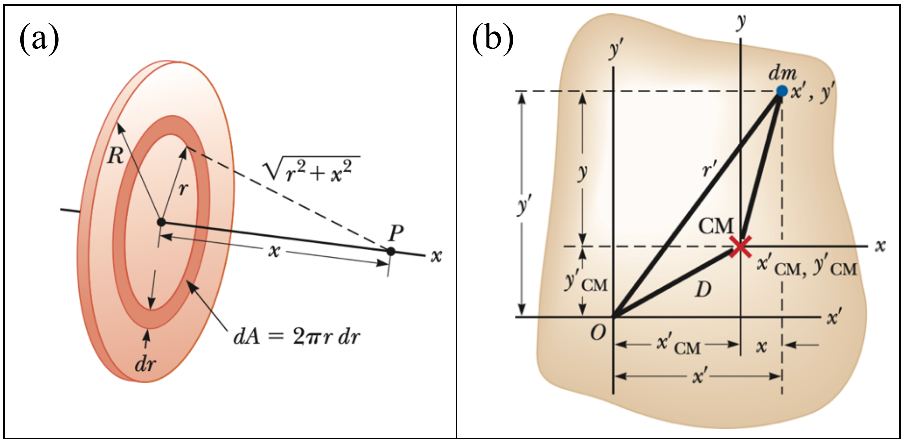


**Figure 2: Differential as (a) small amount quantities and (b) discrete punctual quantities (Serway & Jewett, 2019, pp. 649 and 266)**

## SUMMARY AND SHORT DISCUSSION

Our explorative study of one macroeconomics, one chemistry, and one physics textbook indicates that some of the conceptions of differentials that are commonly used in economics appear in multiple disciplines. For instance, we found the conception of the differential as a small change or increment in all the analyzed textbooks. However, others seem to be discipline specific, like the discrete interpretation of differential quotients in economics or the differential as a punctual quantity in physics. Hence, more data is needed to find out to what extent our preliminary findings from one textbook for each discipline are generalizable.

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# Integrating Computational Thinking into Undergraduate Calculus in Mechanics: Insights from Educators and Communities of Practice

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*In this paper we study how physics educators teaching an introductory course in mechanics address programming and computational thinking in an effort to give students insights into the calculus of the equations of motion. We investigate this through the theoretical lens of Communities of Practice. The theory's three key components domain, community and practice are used as a backdrop when analysing transcripts of semi-structured interviews with three physics educators involved in the introductory course. Our analysis shows that the educators invoke modelling processes by working with the equations of motion numerically in an experimental setting in an attempt to promote students' conceptual understanding of core concepts. We argue that computational thinking can indeed be a helpful tool for students working in the undergraduate introductory course in mechanics, by bridging the components of physics and calculus.*

*Keywords: computational thinking, mechanics, conceptual understanding, communities of practice, modelling.*

## BACKGROUND AND RESEARCH QUESTION

The aim of this research paper is to explore the link between the three components physics (more specifically mechanics), calculus and computational thinking. This is done by investigating how Computational Thinking (CT) is used as a tool when working with calculus concepts in an introductory physics course. The term CT was popularized in a short article by Wing in 2006, but the term has a long history stemming from the constructionist research community of Papert et. al, however with slightly different meanings (Wing, 2006, Papert, 1980, Papert, 1991). A fully agreed upon definition of CT does not exist, but CT refers to a competence that is wider than merely coding or programming. Cuny et al. define computational thinking as "… the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent." (Cuny, Snyder, Wing, 2010, cited in Wing, 2011, p. 20). This definition is also close to the definition given by Denning and Tedre (2019, p. 15): "Computational thinking is the mental skills and practices for designing computations that get computers to do jobs for us and explaining and interpreting the world as a complex of information processes."

We argue that computational thinking goes hand in hand with calculus, which in a historic perspective was developed from a need to be able to make models of what is observed in the real world, and to predict what would be the consequences of this mathematical modeling process. The fundamental tools of calculus are important to all STEM-disciplines, but it is not necessarily easy for students to see the connections between different disciplines.

Orban & Teeling-Smith (2020) propose four rationales for integrating computational ideas into an introductory physics course:

1. CT is integral to physics as a discipline.
2. Using CT to simulate physical systems is often non-trivial.
3. Creating and debugging programs modelling physical phenomena is challenging.
4. CT is valuable as a representation of physics.

With these arguments as a foundation, we study an introductory physics course to see whether we can identify CT's role when working with calculus and the equations of motion. Our research question is:

***"How is the role of computational thinking addressed in the practice of a specific introductory mechanics course when working with the equations of motion?"***

The research setting for this project is an introductory physics course in mechanics intended for first year undergraduate students at the University of Oslo. The course is designed in such a way that the students meet realistic physical modelling using data from their own measurements during experiments. Around 150 students participate in the autumn's run of the course, and 50 students in the spring. The course has six hours of teaching per week. This includes two hours of lectures and two hours of seminars, group work for all participating students with multiple instructors. The last two hours have alternated between traditional work in small groups and the seminar model, as the groups suffered low attendance numbers.

We ground our data generation for this short communication in a document from the course's curriculum: *Experiment 1*. The document describes one of the experiments that are mandatory for the students to participate in to pass the course. The document is used as a foundation because it encompasses CT, physics and calculus in a natural way. A quote from the purpose of this experiment is given underneath to illustrate this point (freely translated from Norwegian to English):

> *Experiment 1* – acceleration: "(…) You will then program a numerical method for integration and use this method to find relevant measures and also discuss the uncertainties in your own results."

We note that both the teaching material and the educators talk about numerical methods in a way that we interpret as part of the definition of CT as defined by Denning and Tedre (2019).

## THEORETICAL FRAMEWORK

The theoretical framework Communities of Practice (hereafter referred to as CoP) forms the backdrop of this investigative project. CoP provides a lens through which the integration of programming into the teaching of calculus in a university physics course can be explored. Learning in CoP is conceptualized as a social process situated within a community that shares a domain of interest, engages in mutual participation, and develops shared practices (Wenger, Trayner & De Laat, 2011). From this we can identify the theory's three main components: domain, community and practice.

*The domain* refers to the area of knowledge that defines the community and gives it its identity. Shared interests and goals are also part of a community's domain. The domain defines the key issues that the community's members address and work with (Smith, Hayes & Shea, 2017). *The community* refers to the individuals who find relevance in the domain and work together with the ideas, goals and topics contained within it. The relationships among members of the community and the boundaries defining the inside and outside of the community are also of key interest. *The practice* refers to the constructs, both cognitive and physical, that enable the members of the community to act in the world (Consalvo et al., 2015). The practice is developed by and shared among the members of the community. The constructs constituting a community's practice consist of, but are not limited to, methods, tools, documents, programs and the body of knowledge constituting the community's domain. In this project we define the community to be the collection of people involved in the introductory mechanics course that we base our data generation on. This involves both employees of the physics department involved in the course (lecturers, administration, course coordinators, etc.) and the students taking the course. When the community is now defined, the domain and practice are more or less self-explanatory. The domain in this project is the body of knowledge contained in the course, both the knowledge that is to be taught during the course and the pre-requisites needed to take the course. Finally, we can deduce that the practice is the collection of constructs associated with the course and its domain. From these definitions we can formulate that the research in this project concerns the lecturers and students of the community, namely the integration of the equations of motion from the domain, and the associated constructs from the practice. We study the practice from the perspective of three educators who are teachers of the course.

## METHODOLOGY

Semi-structured interviews of three educators (E1,E2 and E3) involved in the course were performed. All three educators have been teaching the course, but at different times. One of the educators taught in the spring semester, and the two others in the autumn semester. In the interviews we focused on the role of computational thinking in the teaching of the course. The educators were all asked to reflect on how programming and computational thinking is utilized in their teaching, before the attention was turned towards the experiment and the computational aspects of working with the equations of motion.

As mentioned previously, the interviews are to some extent grounded in one of the experiments that the students take part in during the semester. In this experiment the students use the built-in accelerometer in their own mobile phones as a measuring device. The students measure the acceleration $a(t)$ of a lift going to the top of the physics building, ultimately for finding an estimate of the height of the building. The experiment involves modelling, calibration, measurement and analysis. The measurements give the students a set of discrete data points for acceleration versus time, which must be numerically integrated (using finite differences) to get the velocity $v(t)$ and further integrated to get the position $p(t)$. The results are shown as plots of $a(t), v(t)$ and $p(t)$.

## FINDINGS

The course has been designed to be a first semester physics course, where students meet mechanics and modelling. Most of the students follow the course in parallel with an introductory calculus course at the mathematics department. This means that some of the more advanced concepts of calculus (i.e. differential equations) are not known to the students prior to the physics course.

> E3: "I see that many find it a bit overwhelming, it's a bit much, maybe too many things to master, and it takes some time before they feel confident with it."

Although this could cause some problems, our analysis shows that the educators indicate that by participating in the modelling process as part of the course's shared repertoire, and making their own measurements in these processes, the students get hands-on experience which can be used to build a deeper knowledge. This implies that involving modelling processes as a part of the indigenous purpose of the course's joint enterprise might have beneficial consequences.

Approaching the equations of motion in physics problems through experiments that require measurement and numerical methods gives students a direct way to tackle more complicated problems than they would be able to handle if they were supposed to solve analytical problems.

> E3: "We use numerical methods while they are learning mechanics and simultaneously learning about the analytical solution of differential equations." "They get a more formal introduction to this later in a [calculus] course, ..., we've made it more practical, how to do it both analytical and numerical throughout the semester."

Formulating and solving the numerical problems seems to be easier for the students than finding analytical solutions to the problems, where algebraic manipulations might act as barrier for conceptual understanding. This illustrates how the educators consciously address the shared repertoire of the practice through computational thinking in an effort to lower the degree of abstraction related to the various calculus concepts involved in the equations of motion. The students are in *Experiment 1* facing the fact that small errors in the calibration of the device can lead to large errors when integrating for finding the height of the building. Thus, by implementing modelling and programming as part of the course's domain, the students not only gain experience with how calculus can be used for analysis of movement, but they also get knowledge which is in the core of the game of experimental physics, namely modelling and error estimation.

## CONCLUSION AND FURTHER RESEARCH

The findings in this research project support that computational thinking can be effective as a link between calculus, physics and the technical components involved when working with the equations of motions in an experimental setting. Further research involves going deeper into the analysis of the interviews that has been performed, and doing content analysis on the teaching material used in the course. This would enable us to see if the findings from the content analysis are consistent with the ones laid out in this project.

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# Interpreting simulations of a physical phenomenon: insights from an eye-tracking study

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*The conceptualization of physical and dynamic phenomena through functional relationships requires covariational reasoning to grasp the mutual relations between the quantities involved. Additionally, interpreting and explaining these phenomena necessitates engaging with and linking their representations to achieve a comprehensive understanding. In this contribution, we show how a student engages in covariational reasoning while interpreting and connecting different representations of a physical phenomenon. The methodological tool of eye-tracking was used to collect data on these aspects. The qualitative analysis revealed specific connections between the dynamic object and the graphical representation.*

*Keywords: covariational reasoning, semiotic representations, physical phenomena, eye-tracking.*

## INTRODUCTION

Mathematical reasoning is vital for interpreting dynamic phenomena in the physical world. One of the key forms of reasoning for this purpose is covariational reasoning, which refers to the cognitive ability to understand how quantities change in relation to each other (Thompson & Carlson, 2017). Covariation involves tracking changes in one variable as they correspond to changes in another, such as how distance changes over time or how gravity force changes with respect to the distance between the masses involved. It is fundamental for meaningfully understanding functional relationships in various contexts, including physical phenomena such as motion or growth, economic trends, or ecological systems. The conceptualization of dynamic situations from the physical world requires a suitable form of reasoning to conceptualize both variations and co-variations of the quantities involved. Additionally, interpreting and explaining these phenomena necessitates engaging with their representations and linking different representations to achieve a comprehensive understanding of the phenomenon.

For researchers, accessing students' cognitive processes is nothing but trivial. To achieve this, appropriate methodological tools must be utilized, and effective task designs developed. Eye-tracking technology has proven to be a powerful method for inferring cognitive processes based on eye movements. Literature suggests that it is essential to elaborate on domain-specific interpretations and avoid relying solely on gaze (Schindler & Lilienthal, 2019). In this contribution, we discuss how a student engages in covariational reasoning as he conceptualizes a physical phenomenon by connecting two distinct representations provided.

## THEORETICAL FRAMEWORK

Covariational reasoning entails envisioning how quantities change in relation to each other. It enables individuals to conceptualize the relationship between two quantities by identifying how they change simultaneously (Thompson & Carlson, 2017). For example, let us imagine a ball rolling down an inclined plane. A person being able to reason covariationally would conceptualize how the distance travelled by the ball increases over time. Thompson and Carlson (2017) have developed a taxonomy of six levels of covariational reasoning: no covariation, pre-coordination of values, gross coordination of values, coordination of values, chunky continuous covariation, and smooth continuous covariation. Such levels are descriptive of a person's ability to engage in covariational reasoning. Effective forms of continuous covariation emerge only at the final two levels of the framework, each characterized by distinct underlying images of change—either in discrete chunks or as a smooth progression.

Covariational reasoning is essential for interpreting graphs, equations, or physical phenomena that represent these functional relationships. Moreover, covariational reasoning closely relates to the ability to navigate multiple representations to fully understand dynamic relationships (e.g., Rolfes et al., 2022). Simulations of physical phenomena are represented through various semiotic registers, including graphs, symbolic equations, tables, and verbal descriptions. For instance, to understand the acceleration of an object, one might interpret the slope of a velocity-time graph, write the corresponding derivative equation, and provide a verbal description of the process. Duval (2006) argues that effective mathematical reasoning requires both treatments (manipulations within a single representation) and conversions (connections between different representations). In our view, covariational reasoning may depend on both processes—manipulating individual representations, creating connections, and integrating insights from multiple representations to achieve a coherent understanding.

## METHOD

This contribution analyses an experiment involving a 16-year-old student -Yuval- from a public school in Israel. The study presented here is part of a larger project in which participants engaged with five task units. In this analysis, we will focus solely on the first task unit, which simulates a physical phenomenon. The first animated task tackled the phenomenon of a ball rolling down an inclined plane (the well-known Galileo experiment), focusing on the dependence of the traversed distance on time. The animated task consisted of three parts and here we will present the first two: (a) a short *dynamic stimulus* consisting of a simulation of the physical situation (https://www.youtube.com/watch?v=U2XfNJCJdkw) which was shown three times; (b) a *descriptive task*, where, starting from a picture of the inclined plane on the left and the distance-time graphical representation on the right, the student is asked to describe the mutual dependence of time and distance (Figure 1) and so possibly engaging in forms of covariational reasoning.

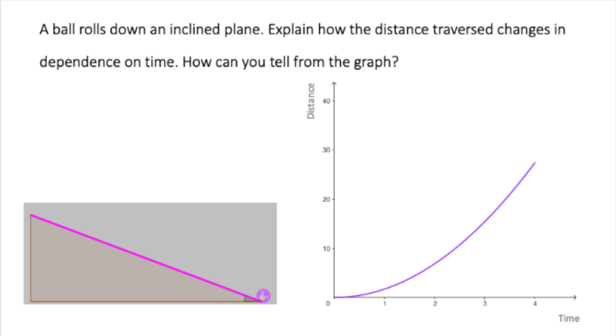


**Figure 1: Descriptive task (translated from Hebrew into English) with the inclined plane on the left and a graphical representation on the right.**

While the entire experiment was video-recorded, gaze-related data was collected with a screen-based eye tracker: Tobii Pro Fusion (250 Hz, binocular). A 9-point calibration procedure was performed, and inaccuracy was considered in the task design and data interpretation. After solving all the task units, a stimulated recall interview (SRI) was conducted: the student was shown the gaze-overlaid video generated by the software Tobii Pro Lab as stimulus. In such a video, gazes are displayed as animated wandering red dots and connected by lines helping the observer to keep track of their sequence. During the SRI, under the interviewer's (the third author) questions, the participant could provide an explanation for his gazes and his claims. No questions were asked when re-watching the stimulus (a) phase. In this contribution, we qualitatively analyse the piece of experiment concerning the dynamic stimulus (a) and the descriptive task (b). The analysis was conducted by considering both the videorecording of the experiment and the gaze-overlaid video which were integrated and analysed with the support of Camtasia Studio software. The analysis focuses on the relevance attributed by the student, and detected by interpreting his gazes, to the various representations at stake. An interpretation of the emerging covariation was elaborated based on the gaze plots, and the student's claims (during the experiment and the SRI).

## RESULTS

This episode, which lasts in total 56 seconds, illustrates how Yuval employed covariational reasoning to understand the dynamics of a ball rolling down an inclined plane. Analysis of Yuval's gaze indicates that he spent the first 15 seconds exploring the relationship between two representations: the dynamic simulation and the graphical representation. Shortly afterwards, he read the verbal instructions for the task, which took him about 14 seconds. From the second 29 until the end of the episode (the descriptive task), Yuval focused solely on the graphical representation.

Yuval's initial focus was on the label of the vertical axis (s-axis from now on), where the word "distance" is displayed. He stared at this word for 1.14 seconds. After that, he directed his gaze toward the middle of the inclined plane (dot 1 in Figure 2a) for 0.10 seconds. Once the ball began to move, Yuval followed its trajectory, looking at three different locations (dots 2-3-4 in Figures 2a, b, and c). His fifth gaze (dot 5 in Figure 2d) anticipated the ball's movement along the inclined plane, suggesting that Yuval was predicting where the ball would go next. In other words, Yuval's gaze followed the ball's positions, suggesting that he started covarying time passed and the distance travelled. SRI could not help to validate this interpretation.

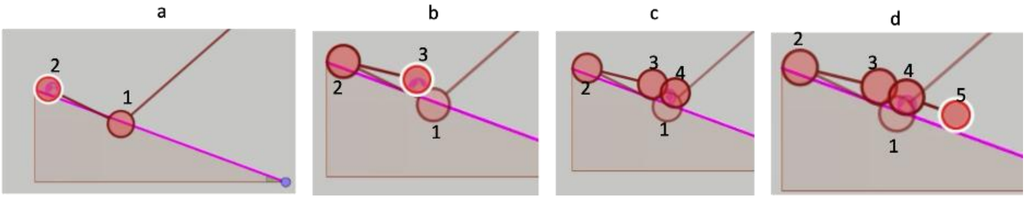


**Figure 2: a) Yuval looks first at the center of the inclined plane (dot 1) and then at the ball as it moves (dot 2); b) and c) Dots 2 and 3 suggest Yuval was focused on the ball; d) In dot 5, Yuval anticipates the ball's movement.**

At second 3.17, Yuval shifted his gaze to the graphical representation on the right. First, he focused on the middle of the graph (Figure 3a), where his fixation lasted for 0.10 seconds. He then moved his gaze to the far right of the graph, maintaining his focus there for 0.50 seconds. At this moment, the simulation replayed, and Yuval redirected his gaze to the last part of the inclined plane, which he observed for 0.70 seconds. Shortly thereafter, he shifted again to the graphical representation. This rapid switching between the two representations suggests that Yuval understood the dynamics of the ball but still needed to interpret the graphical representation and so enact a conversion. At second 7.12, while watching the stimulus for the third time, Yuval followed the time-distance graph with his gaze from start to finish (Figure 3b). Shortly after that, he focused his attention on the label of the s-axis (Figure 3c). At second 15.26, Yuval read the task instructions for approximately 14 seconds.

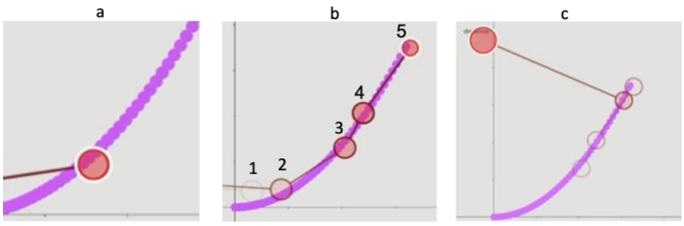


**Figure 3: a) Yuval focused on the middle of the graph; b) followed the time-distance graph; c) focused on the s-axis label.**

At 29:06, he focused his gaze first on the s-axis label, then on the horizontal axis (t-axis from now on) label (time), repeating this movement twice. While examining the axes' labels, Yuval concluded, "Okay, the graph represents the distance-time relationship." As he pronounced the word "time," he shifted his gaze along the t-axis from right to left before returning to the s-axis label. The following sentence, "As time passes, the distance increases", can be interpreted as an expression of the student's covariational reasoning, and in particular a gross coordination given that he qualitatively expresses the co-variation of the two quantities. However, the analysis of the student's gaze plot, which follows the entire graph, and the subsequent verbal explanation, suggests a smooth continuous covariation. Yuval was not only satisfied with such a qualitative covariational description of the graph but added justification for why the graph is a curve, "because the ball accelerates," so engaging in a conversion between the graph and the phenomenon simulation. He then looked at several values on the s-axis, moving from the bottom to the top. He continued, "We can see that the graph is increasing, but not in a straight line," and followed the graph with his gaze in the same manner as shown in Figure 3b. Such elaboration was confirmed by the SRI.

## CONCLUDING REMARKS

This short episode, taken from a longer experiment and a larger project, aims to explore how students use calculus concepts, such as covariation, to interpret physical phenomena, utilizing eye tracking as a methodological tool. The study shows how a student engages with multiple representations (a dynamic simulation, a static representation and a graph) to grasp the functional relationship between time and distance as a dynamic relationship. In the case discussed here, the student began his exploration by focusing on the provided dynamic simulation to understand the graph representing the motion of a ball. Through the analysis of the student's gaze, a connection was revealed between the dynamic object and the labels on the axes of the Cartesian coordinate system. In addition, our analysis revealed that the analyses of covariational reasoning through verbal language or eye movements are non-redundant with respect to each other. This confirms that such semiotic resources cannot be considered in isolation to achieve a full understanding of the investigated cognitive process (Bagossi et al., accepted).

## DECLARATIONS

The Human Subjects Research Committee of Ben-Gurion University has approved this study. The authors thank Maike Vollstedt and Aylin Thomaneck (University of Bremen, Germany) for contributing to the design of the experiment.

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# LLMs as world builders for authentic problems in calculus for STEM students

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*Developing real world problems in Mathematics for tertiary students of other disciplines requires some knowledge in the other discipline. What is an authentic problem in this context, and can an LLM (Large Language Model) be an effective helper in this task?*

*Keywords: Mathematics for other disciplines, reality problems, Artificial Intelligence.*

## SCI-PROBLEMS

Let us consider a scientific discipline, which we will call $S$, different from Mathematics; and suppose that we want to teach a subset $M$ of the mathematical world to students whose main interest lie in $S$. In order to avoid disengagement, we want the students to perceive a utility-value of $M$ through modelling in $S$. *Mathematical modelling modelling in the discipline* $S$ is a "process consisting of structuring, generating real world facts and data, mathematising, working mathematically and interpreting/validating" (Niss et al., 2007). We will be interested mainly in the map $f: S \to M$, both since it appears to be the most relevant in a Mathematics class, and by analogy to translators, who usually work translating *into* their mother tongue and not *out of*.

Carotenuto et al. (2021) define word problems as a literary genre characterised by:

I) a narrative component that introduces and describes the context and the characters;

II) an informational component giving the information needed to solve the problem;

III) the question component.

This is an inclusive definition, which includes real life problems (including most mathematical modelling), as opposed to the one of, say, Kaiser (2017) where word problems are defined as those whose solution is only important in the school context.

We now define a *sci-problem* $S \to M$ as a word problem relative to $S$, which needs tools from $M$ to be solved.

A sci-problem could be a veil in front of a standard calculus exercise where the map $f: S \to M$ is a matter of a "simple" semiotic transformation (Figure 1) or it could be a problem (Figure 2) where the mere definition of the map $f$ would be both problematic and most likely different between members of the $S$ community of practice and members of the $M$ community of practice (Rogovchenko & Rogovchenko, 2023).

## A CLASSIFICATION OF SCI-PROBLEM

Science-fiction is another literary genre related to science, and their relationship can be classified on the spectrum of *hard* and *soft* science-fiction.

Hard science-fiction might be defined by “its relationship with science” which “provides the illusion of both realism and rationalism.” It’s a sub-genre in which authors “need not to think [the science] likely to be true; only that it should not compromise fidelity to facts for purposes of narrative drama or, if it does, that this should happen in such a way that the reader does not much notice” (Cramer, 2003).

Soft science-fiction, on the other hand, is “science-fiction in which the scientific elements are relatively unimportant to the story” (“Soft Science Fiction”, 2007).

> The concentration $C(t)$ of a drug in the bloodstream (measured in mg/L) as a function of time $t$ (in hours) after administration is given by the formula:
>
> $$C(t) = 50te^{-0.2t}, \qquad t \geq 0.$$
>
> 1. Find the time $t_{\text{max}}$ at which the drug concentration reaches its maximum value.
> 2. Compute the maximum concentration $C_{\text{max}}$.
> 3. Discuss the practical meaning of $t_{\text{max}}$ and $C_{\text{max}}$ in the context of drug administration.

**Figure 1: Soft sci-problem [1]**

> Driving across Nevada, you count 97 dead but still easily recognizable jackrabbits on a 20 km stretch of Highway 500. Along the same stretch of highway, 28 vehicles passed you going the opposite way. What is the approximate density of the rabbit population to which the killed ones belonged?

**Figure 2: Hard sci-problem (Harte, 1988)**

## Hard sci-problems

A *hard sci-problem* will be a sci-problem where science provides the illusion of realism, the setting should not compromise fidelity to scientific facts for didactical purposes or, if it does, this should happen in such a way that the student does not much notice. An example is in Diagram 2: we do not know if the numbers mentioned were collected by an actual field experiment or if they were made up by Harte, what matters is that the student would not notice the difference.

Designing a hard sci-problem $S \to M$ requires knowledge not only of the mathematical content $M$ and of its relative pedagogical content knowledge, but also of $S$. This usually require “the development of interdisciplinary subjects taught by an interdisciplinary team” (King et al., 2023).

A hard sci-problem $S \to M$ will appear, to a student, as a problem in $S$—i.e., a problem that an actual scientist in $S$ might encounter in her scientific endeavour—which requires $M$ techniques to be solved.

### Soft sci-problems

A *soft sci-problem* will be a sci-problem in which the science is irrelevant to the story. In the example of Figure 1 we see a stereotypical scholastic world problem.

Designing a soft sci-problem $S \to M$ does not require much more knowledge about $S$ than being able to read the mappings $S \to M$ and $M \to S$: in the case of a first-year calculus class, this means not much more than what is taught in high school.

A soft sci-problem $S \to M$ will not appear, to a student, as a problem in $S$ but a way to make an $M$ problem somewhat relevant; in many cases, *suspension of sense making* (Carotenuto et al., 2020) will appear and students will have a hard time on the $M \to S$ mapping (see point 3 in Figure 1).

## LARGE LANGUAGE MODELS

> Large Language Models (LLMs) are advanced artificial intelligence systems trained on vast amounts of textual data to process and generate human-like language. They operate by predicting the most probable sequence of words, enabling applications such as text generation, summarization, and question answering. In education, LLMs offer opportunities for personalized learning, automated content creation, and enhanced interactivity, reshaping traditional teaching and learning paradigms. [2]

Despite the shortcomings of LLMs—hallucinations being a major problem [3]—ChatGPT has adequate abilities in semiotic transformations, including the creation or adaptation to different cultural contexts of simple world problems (Kwon, 2024).

ChatGPT 4o is similarly adequate in creating simple soft sci-problem $S \to M$ as long as the user has a cursory knowledge of $S$:

> User: I need to write an exercise for students in my Calculus course where they calculate the maximum of a function on $[0, +\infty)$, but the function should be presented as the concentration of a drug in the bloodstream.
>
> Assistant: Here's an example of an exercise you can propose: [text with parameters]
>
> User: Add specific values.
>
> Assistant: Here's the exercise rewritten with specific values: [text of Figure 1]

On the other hand, ChatGPT *might* be able to create a hard sci-problem $S \to M$, but writing the necessary prompts requires a knowledge of $S$ like the one required to design such a problem.

## BUILDING WORLDS

The creation of (alien or futuristic) world is a big component in hard sci-fiction: authors have to devote themself to the creation of an ecology, an institutional and a technological framework, possibly even a language. A book writer might spend years of research building such a world. Similarly, a calculus instructor $x$ might spend years studying for a Bachelor in $S$ and then be ready to write a hard sci-problem $S \to M$: in most cases, this is completely unfeasible, and she would need to rely on help from a colleague $y$ in $S$, with both willing to spend enough time to develop a translation framework $S \to M$ and $M \to S$. Using the ATD (Anthropological Theory of the Didactics) framework, $y$ is helping $x$ attaining $O$, i.e. the content knowledge of $S$ required to be an adequate designer of hard sci-problem $S \to M$.

Our research question is then: can an LLM play the role $y$? That is, can an LLM help $x$ investigate existing $S$-problems $A^\lozenge$ and build an appropriate hard sci-problem $A^\heartsuit$?

The answer to the question is yes: ChatGPT 4o can be an effective helper as long as the user $x$ is able to *convince* the LLM $y$ to be of help. Indeed, $y$ does not seem able to generate directly anything more than a soft sci-problem, but it is able to look for relevant references and explain its content (see Rizzo (2024) for the whole transcript).

User: Now I need to write a problem, still about mathematics, for my students. I'd like to use one of the models used in botany to connect leaf inclination to the amount of solar energy absorbed. Can you help me?

Assistant: Certainly! In botany, the inclination of leaves and the amount of solar energy absorbed can be modeled using mathematical concepts that relate angles to light intensity, often relying on the **cosine of the angle of incidence** between sunlight and the leaf surface. Below, I'll suggest a model and a problem for your students.
[Produces a soft sci-problem with detailed solutions]

User: Do you know of any references in the literature to problems of this nature?

Assistant: Certainly! The orientation of leaves, known as **Leaf Angle Distribution (LAD)**, significantly affects the absorption of sunlight by plants. Mathematical models describing this relationship are crucial for understanding and predicting the photosynthetic efficiency of plant canopies.
[Proceeds with an explanation of Leaf Angle Distribution, and presents a couple of *real* relevant references]

In the first interaction $y$ does not help $x$ by producing no relevant answer $A^\lozenge$. A second interaction, on the other hand, goes in the right direction: $y$ produces relevant answers (there are mathematical models on the leaf angle distribution), explains enough of $S$ with a language comprehensible to $x$, and offers resources for additional exploration of relevant cases of $A^\lozenge$. Now $x$ has to read the resources (eventually going back to $y$ for help understanding a concept in $S$) and use his Pedagogical Content Knowledge (PCK) to isolate a problem which is relevant to his goals.

## CONCLUSION

ChatGPT 4o can be effectively used to help in the design of modelling problems in a scientific domain, say botany, essentially unknown to a calculus instructor.

This requires, though, the instructor to possess a sufficient level of Technological Pedagogical Content Knowledge (TPACK) to interact in a fruitful way with the LLM; and of PCK to recognise in the produced botanical literature elements conductive for a modelling problem relevant to his didactical goals.

## ACKNOWLEDGMENT

This research has been partially supported by GNSAGA–INdAM.

## NOTES

1. Text generated by ChatGPT 4o
2. Text generated by ChatGPT 4o
3. E.g., I was not able to get ChatGPT to generate an adequate text to the previous note with proper references to *existing* literature.

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# MAPLE LEARN: INVESTIGATION OF STUDENTS' EXPERIENCE IN LEARNING THE CALCULUS COURSE IN ONTARIO

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*This paper aimed to investigate the impact of incorporating digital technology, specifically Maple Learn***,** *on the development of mathematical thinking in a calculus course offered by a large university in Toronto. Through activities with four participants, we explored how Maple Learn's functionalities support learning in undergraduate calculus, addressing a gap in existing research. We discussed three key objectives: exploring Maple Learn's functionalities for mathematical thinking, examining the relationship between these functionalities and the context of calculus, and analyzing the experiences and perspectives of learners who integrate Maple Learn into their practices to enhance engagement and understanding of calculus, both in academic and real-world contexts*.

*Keywords: Calculus, Maple Learn, Mathematics Education, Functions.*

## INTRODUCTION

The rise of new digital technologies has led to new ways of engaging with mathematics both as a scientific discipline and as a school subject (Hoyles, 2018). Key to understanding these new ways is the development of mathematical thinking – the cognitive process of understanding, analyzing, and synthesizing mathematical concepts and processes to solve problems and make informed decisions (Schoenfeld, 1994). This thinking involves logical reasoning, critical analysis, and the ability to generalize and adapt concepts to various contexts. Integrating digital technologies requires a shift towards strategies that encourage understanding the world through a mathematical lens (Jarvis et al, 2022).

In this project, we will examine the use of digital technology (mathematical software) for the development of mathematical thinking. While mathematical software provides the means to carry out operations, mathematical thinking is the overarching skill that empowers individuals to grasp the essence of mathematical concepts, approach problems creatively, and select and apply the appropriate formulas and procedures in a context to develop innovative solutions. While it is clear that students can benefit from access to instruments that effectively promote mathematical thinking through mathematical software, little research has examined the technical functionalities of such platforms.

This paper also investigates the calculus across the contexts in which mathematical thinking is developed. In broader terms, we differentiate two contexts of calculus: formal and non-formal. Formal promotes the study of mathematical theories and structures, typically employing axiomatic systems, formal proofs, and symbolic notations. Whereas, non-formal encompasses that emphasizes the ability to interpret, analyse, communicate, and apply basic mathematical concepts and skills to solve everyday problems. In light of these considerations, this research delves into the experiences of learners as they navigate the development of mathematical thinking through the calculus course offered in their undergraduate programs, contributing valuable insights to the broader discourse on digital technology's role in calculus across different contexts.

## CONCEPTUAL FRAMEWORK

The conceptual framework of this research is supported by the multifaceted nature of the calculus I course offered by a large university in Toronto, Ontario (Canada). This serves a wide range of purposes, including reinforcing high school mathematics. Many high-school math courses (even Advanced Placement and other "university-level" courses) focus on memorization and computations. By contrast, university courses focus on understanding why methods work and what the concepts mean. More emphasis on understanding over memorization to sharpen students' problem-solving skills. The central focus of this study is the incorporation of technology in Calculus I, a course offered in the first year of undergraduate programs at the university, to support the development of mathematical thinking. It also explores the experiences and perspectives of students who use technology to learn various Calculus concepts. In this course, students gain a deeper understanding of how to apply Calculus concepts to the social, biological, and physical sciences. More broadly, we distinguish four contextual domains for this Calculus I course: Social Science, Applied Science and Engineering, Hard Science, and Life Science. Hard science we considered under a formal calculus context. Social science, Applied science and Engineering, and Life science are considered under a non-formal calculus context. Unsurprisingly, calculus in these four contexts receives little attention in research regarding the integration of digital technologies. This study focused specifically on both formal and non-formal calculus contexts, delving into the realm of mathematical thinking and examining how technology (Maple Learn) integrated it into the calculus for the first-year undergraduate university students. Additionally, we highlight the need for a systematic framework to understand each context that offers unique mathematical ideas and representations, giving rise to distinct forms of mathematical thinking, which the project seeks to explore and understand the affordances and obstacles of Maple Learn in these courses. Finally, the project can help improve the quality of mathematics education by providing evidence-based recommendations on the use of mathematical software in mathematics classes.

## CONTEXT AND METHODS

To answer the research questions, the method employed in this paper involves offering activities based on Maple Learn and the exponential function. The main objective of the study was to assess whether students' problem-solving skills can be enhanced with the help of Maple Learn. The exponential function, a topic students have already learned in high school, was chosen to focus on understanding why methods work and what the concepts mean. The activities were conducted over three days, with one hour allocated daily. On the first day, students were introduced to Maple Learn, learning how to access and use the software. On the second day, they worked on an exponential function activity using paper and pencil. On the final day, they completed the same activity with the help of Maple Learn. The activities explored various aspects of Maple Learn's functionalities, focusing on how they support mathematical thinking, their relationship to calculus concepts, and the experiences and perspectives of learners using the software to enhance engagement and understanding of calculus in both academic and real-world contexts. AS this study is in its initial stages, it was conducted with four students from the Calculus I course. We asked the students to describe how they explored problems involving exponential functions using Maple Learn. Each student submitted a work file that they used during problem-solving with the help of Maple Learn. An exploratory causal path analysis was performed to assess the effect of the students' use of Maple Learn functionalities on their cognitive processes specifically, their ability to understand, analyse, and synthesize mathematical concepts and procedures to solve a problem and make informed decisions (Jo¨reskog & So¨rbom, 1985).This paper focuses on analyzing two sets of questions related to: 1) How well does Maple Learn align with its intended goals for use in calculus I classes? 2) Students' understanding of calculus topics and the software's effectiveness in enhancing their learning. It seeks to answer the following research question: *How can mathematical software support the learning and understanding of concepts in Calculus I?*

## FINDINGS

The first research question examined whether students utilized Maple Learn's functionalities to support mathematical thinking, particularly within the context of Calculus problem solving. Maple Learn offers five key functionalities designed to align with various levels of mathematical understanding: analyzing, planning, and verifying, among others. Analysis of student activity revealed that participants engaged extensively with the platform's mathematical outputs and operations. To address the second research question, a control group of students participated in a three-day workshop focused on application problems involving exponential functions. Across three structured activities, students engaged with three distinct problems related to this topic. During the workshop, students recorded the problems they encountered, the strategies they employed, and the solutions they independently derived. These records served as a rich source of data reflecting students' learning experiences and their evolving understanding of exponential functions. Descriptive analysis of the data suggested that students responded positively to the integration of numerical and visual elements within Maple Learn. These features appeared to enhance their conceptual grasp of the mathematical content. Furthermore, students' work demonstrated improvements in their ability to analyze problems and devise effective solution strategies. The findings suggest that engagement with Maple Learn supported the development of analytical skills and fostered a stronger capacity to identify and implement appropriate solution pathways.

In conclusion, the study indicates that the use of Maple Learn contributed to increased student engagement and active participation in the learning process. Students not only attempted to replicate graphical representations but also demonstrated heightened interest in exploring the properties and applications of exponential functions. Moreover, the platform promoted collaborative learning, as students frequently engaged in reviewing and analyzing one another's work. This collaborative process provided valuable opportunities for peer learning, critical thinking, and deeper conceptual understanding.

## DISCUSSION AND CONCLUSION

The results of the paper highlight a notable trend: the integration of technology, such as Maple Learn, within calculus courses is crucial for developing mathematical thinking. As society becomes increasingly quantitative, problem-solving situations demand a broader familiarity with mathematics. Mathematical thinking involves adopting a perspective that views the world through a mathematical lens and seeks logical explanations. Students who grasp the elements of mathematical thinking can independently apply these skills to understand the mathematics they encounter. In the Greater Toronto Area (GTA), post-secondary education systems impose tight time limits on completing mathematics tests, encouraging fast-paced work. However, real mathematics takes time, and much of the process may not initially seem like doing math, as the focus is on thinking mathematically rather than merely applying standard techniques to solve problems. The paper suggests that integrating technology like Maple Learn into Calculus I enables students to build a strong foundation in applying calculus concepts across various contexts. By fostering mathematical thinking, this approach encourages deeper understanding and helps students learn through reasoning rather than relying solely on rote memorization in their calculus classes.

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# MATH-CHIAVELLI: DESIGNING A CALCULUS CARD GAME

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*This work presents the genesis of Math-chiavelli, a card game designed to support university students who struggle with calculus, particularly in "weak" mathematics courses such as Geological Sciences. The design choices are articulated through the framework of mathematics education theory to enhance students' understanding of characteristics and graphical representations of real-valued functions of a real variable. By facilitating the translation between different semiotic representations, Math-chiavelli encourages strategic decision-making, peer feedback, and collaborative learning. This game-based approach aims to build coherence in students' mathematical reasoning, while making calculus more accessible and engaging.*

*Keywords: calculus, semiotic registers, game-based learning, peer feedback, graphical representations.*

## INTRODUCTION AND RATIONALE

In Italy, about one in five students drops out of university within the standard duration of their degree, with scientific disciplines often experiencing higher dropout rates (ANVUR, 2023). A key factor contributing to this is the mathematics course, a mandatory requirement also in degree programs such as *Geological Sciences* or *Pharmacy*, typically chosen by students who do not have a strong affinity for mathematics. In these programs, the exam syllabus is primarily focused on calculus, a subject many students find overwhelming. Failing the maths exam is a common reason for dropout (Ellis et.al., 2014). Calculus, however, plays a crucial role in a range of scientific disciplines: biologists, chemists, economists, and geologists apply calculus in different ways, using its concepts to model phenomena, reason quantitatively, and support argumentation within their fields. Therefore, it becomes paramount that students in these degree programs who attend calculus courses not only pass their exams but also develop the mathematical reasoning necessary for their future careers. Several studies investigate the use of games for teaching calculus concepts, exploring their potential to enhance engagement and comprehension (Ahmad Bakri et al., 2020; Szilágyi & Körei, 2022). Following this line, a researcher in mathematics education (the first author) and a researcher in mathematical analysis who teaches calculus to Geology students (the second author) joined forces to develop *Math-chiavelli*, a card game designed to make mathematics more accessible and engaging for students who did not choose mathematics as their primary educational focus. The educational objective of the game is to support students in conceptualising and systematising mathematical ideas they are already familiar with but may struggle to apply in practice. It aims to enhance their conceptual understanding of calculus by fostering key skills such as interpreting (i.e., reading functions properties from) and constructing graphs of real-valued functions of a real variable, as well as promoting coherence in reasoning.

## THEORETICAL BACKGROUND

In the research on learning and teaching calculus, several frameworks provide insights into how students can better engage with and understand calculus concepts. Following Bressoud and colleagues (2016), we consider Duval's theory of semiotic representation (1995) as focal to investigate students' cognitive development. Duval's theory focuses on the relationships between signifiers structured into semiotic registers and the mathematical concepts they signify. According to Duval, mathematical understanding is constructed through the interplay of different semiotic registers (verbal, graphical, symbolic, and numerical) each with its own rules and characteristics. Two core cognitive processes are central to this interplay: *treatment* and *conversion*. Treatment refers to manipulating representations within the same register (e.g., solving an equation algebraically), while conversion involves translating between different registers (e.g., moving from a verbal description of a function to its graphical representation). Both processes contribute to learning, but conversions are particularly effective in fostering a conceptual understanding of mathematical concepts, as they enable students to navigate efficiently between different forms of representation.

In addition to Duval's semiotic theory, we build on gamification and peer assessment as theoretical perspectives for designing an educational game. Gamification in education incorporates game design elements such as competition, strategy, rules, decision-making to enhance student motivation and engagement. (Deterding et al., 2011). Peer assessment, where students evaluate each other's work and provide feedback, fosters collaborative learning and self-reflection (Topping, 1998).

Given these theoretical choices, the design process is guided by the RQ: How can we design a card game fostering students' conceptual understanding of calculus?

## DESIGNING THE GAME

We designed *Math-chiavelli* as the mathematical version of *Machiavelli*, a Rummy-derived card game, widely popular in Italy, named after the Italian Renaissance Florentine diplomat Niccolò Machiavelli. The game is conceived to help students to build coherence in their mathematical reasoning, understood as the ability to consistently connect characteristics of functions of a single real variable and in understanding and interpreting the related graphs. Students play independently, fostering autonomy in reasoning, though the game also allows for a follow-up teacher intervention to consolidate learning. Math-chiavelli is designed to be played by 2 to 6 players with two complete decks of 75 cards. The cards feature function characteristics, presented in verbal or symbolic form and divided into five colour-coded categories. The full deck of cards can be seen here; in the cards expressed in a verbal register, it is always implied either *the function has,* or *the function is*.

### Games rules

- The dealer, chosen at random, deals 13 cards to each player in clockwise direction. After dealing, the remaining deck is placed in the centre of the table

- The player to the left of the dealer goes first, and play continues clockwise
- There are four actions that a player can play:
  Action 1: Play a *valid combination* of cards on the table, i.e., *at least three cards with coherent function characteristics that either share the same colour* (Fig. 1a)*, or have all different colours* (Fig. 1b)*, completed by a graph of a function with those characteristics sketched by the player*
  Action 2: Add one or more cards with new characteristics and compatible colours to existing combinations on the table, if the sketched functions satisfy the new characteristics
  Action 3: Remove or rearrange cards from existing combinations to create new *valid combinations* that include one or more cards from the player's hand
  Action 4: Draw a card if no other action is possible and wait for the next turn
- Each player must play at least once action 1 before playing action 2 or 3
- Players may not remove any cards already on the table
- Players who make a mathematically wrong move must draw a card and wait for the next turn
- The player who manages to play all the cards in his hand wins the game.

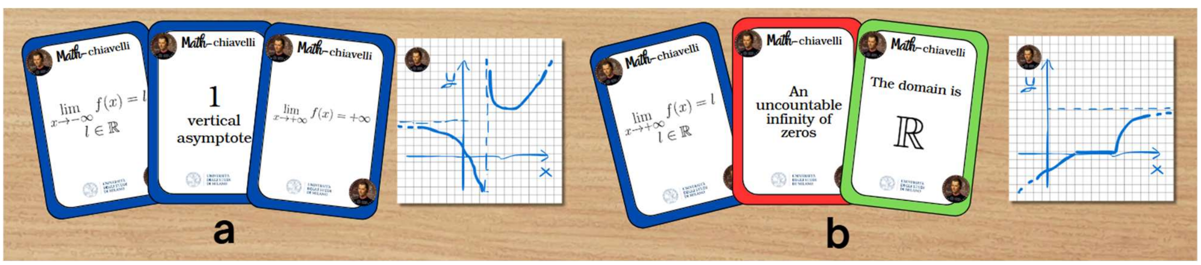


**Fig. 1: Examples of valid combinations**

## Design choices

We wanted to design a meaningful game both from a mathematics education perspective and in terms of the players' experience. This goal guided our design choices, some made a priori, based on the theoretical framework and educational needs, while others emerged a posteriori during play sessions with fellow researchers.

The first a priori choice was about the rules of the game, inspired by the original game with additions aimed at igniting cognitive processes. Constructing a valid combination (action 1) requires players to understand the function characteristics described in the cards and make coherent selections, while the added graph mobilises Duval’s (1995) conversion process from the cards’ verbal/symbolic description into a graphical representation (the sketched graph). This sketching step not only should reinforce students’ understanding but also engage their creativity, as they actively construct a visual representation rather than simply reading properties from an existing graph. Since graph interpretation is already addressed in action 2, we found it more compelling for action 1 to challenge students to invent the graph themselves. Adding to existing combinations (action 2) requires analysing the current graphical representation and determining how a new card integrates into it, which involves converting from the graphical to the verbal/symbolic register. Removing or rearranging cards to form new valid combinations (action 3) involves conversions in both directions, moving between verbal/symbolic and graphical representations to check that the graph satisfies all the conditions in the cards. These rules leverage gamification principles such as strategy, and decision-making to engage students while ensuring that every decision fosters their conceptual understanding of function properties.

A second a priori design choice was whether to require players to produce a function graph, its analytical expression, or both. The decision to focus exclusively on graphs was based on educational needs and gamification principles. For the target audience, creating and interpreting graphs is a critical skill that the game is designed to develop. Moreover, graphs are better suited to the fast-paced nature of the game: they allow for quick evaluation by players and for a teacher's follow-up intervention that can focus on producing the analytical expressions of the functions involved. These design choices imply also that all players check the coherence between a graph and the cards played by others. This process can mobilise peer assessment and, once again, draws on Duval's (1995) conversions between semiotic registers, as players translate mathematical information across different representations to ensure coherence. This aims at deepening engagement with the content, and fostering dialogue and a clearer understanding of errors, ultimately strengthening conceptual knowledge.

Educational needs and gamification principles also shaped the final a priori choice about the card topics and content, covering domain, global properties, zeros, sign, limits, asymptotes, behaviour, extrema and singularities. Concavity and inflection points were excluded to align with the mathematics degree Calculus course syllabus taught by the second author, as they add complexity without significant benefits for the target audience of the course (and game). Card content was selected to widen students' example space (Watson & Mason, 2005) and ensure that each card can combine with some others in the deck to prevent the game from getting stuck. The cards formulations were also carefully designed to prompt conversions between representations and to avoid misconceptions that would be challenging to address without teacher oversight, as students are expected to play independently.

All a-posteriori design decisions aimed to improve the game's mechanics. The first key change was colour-coding the cards into five categories and limiting valid combinations to differently coloured cards. Without this structure, players tended to create overly complex combinations using many cards. Later, we allowed same-colour combinations to promote reflection on mathematical coherence. Finally, introducing a penalty for mathematically wrong moves proved crucial for fostering peer evaluation.

## CONCLUSION

As educators, it is essential to bridge the gap between traditional calculus teaching and how calculus is understood and applied in various scientific disciplines, shifting the focus from mastering abstract concepts to scaffolding learning in ways that are functional to their future professional needs. Going back to our RQ, we believe that adapting a popular game through the incorporation of mathematical content and rules that ignite cognitive processes like Duval's conversions (1995), Math-chiavelli leverages gamification principles to create a low-stakes environment that fosters learning, strategic decision-making, and immediate peer feedback, all while nurturing coherence in students' mathematical reasoning. We are currently experimenting with the game, and we look forward to sharing insights with the CalcConf community

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# Multivariable limits in different university degrees. A first approach through textbook analysis

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*Multivariable limits play an important role in university mathematics courses addressed to mathematicians and non-mathematicians. Using the Onto-semiotic Approach, we carry out a comparative analysis of three university textbooks used in different degrees: mathematics, engineering, and economics. We identify differences in the different primary objects (situations, languages, propositions, procedures, and arguments). Our findings illustrate a rather different treatment of this concept among these degrees and suggest further research in this line.*

*Keywords: calculus, multivariate limit, textbook analysis, Onto-semiotic Approach.*

## INTRODUCTION

Calculus plays a very important role in many different courses addressed not only to mathematicians, but also to other non-mathematics professionals such as economists or engineers (Biza et al., 2022). In fact, "multivariable functions play an essential role in science, technology, engineering, and mathematics (STEM), as well as in other fields of knowledge" (Borji et al., 2022, p. 1). Thus, there is an increasing amount of research about different aspects of the teaching of multivariable calculus and, in particular, of multivariable limits (Martínez-Planell & Trigueros, 2021).

In this context, we can consider the following research question: What are the differences in the treatment of multivariable limits among different university degrees? Here, we make a first approach to this question by restricting our attention to textbook analysis (Son & Diletti, 2017). Furthermore, we focus our research on three different university degrees: mathematics, computer engineering, and economics (Bailey et al., 2024). Thus, our main goal is to analyze the presence and treatment of multivariable limits in textbooks used in each of these university degrees.

## THEORETICAL FRAMEWORK

Textbooks are a very important source of information in the relationship between research and teaching. More specifically, textbook analysis is a relevant topic for both teaching practice and research in mathematics education (Schubring, 2022). In particular, Randahl (2012) points out that there is a need for more research on university level mathematics textbooks.

The notion of limit in textbooks has been studied, for pre-university and university levels. In the case of pre-university textbooks, some authors, such as Arnal-Palacián et al. (2024) for different types of limits — sequence and function in one variable —, point out that they present both definitions and examples, as well as different registers of representation, the predominant one being the verbal register. For calculus textbooks used in university courses, Hong (2023) examined how the notion of limit in one variable is introduced, and what examples and problems are presented. For the particular case of the limit in several variables, Thompson and Wiggins (1990) point out that the presentation of this notion in textbooks has evolved over time. Attention began to be paid to the notion of the limit from 1950. In the 1970s, a good part of textbooks included a formal definition of the limit, and it is from 1980 onwards that many authors chose to treat the limit intuitively in the main body of the text. In all these periods, the more rigorous the definition was, the more examples and exercises were added.

This work is based on the Onto-semiotic Approach (OSA, see Godino et al., 2007). The theoretical constructs elaborated in this model are articulated in practices, objects and processes, establishing six mathematical objects that are considered primary. All of them can be found in mathematics textbooks, and they are: problem situations (tasks, exercises or problems that may be of an intra- or extra-mathematical nature), language (terms, expressions, notations or graphics, in their different registers of representation), concepts (definitions or descriptions), propositions (statements about concepts or properties of the mathematical objects involved), procedures (algorithms, operations and calculation techniques, which make it possible to solve the problem-situation), and arguments (statements that explain and validate the propositions or procedures).

The existing textbook research from the perspective of the OSA is usually oriented towards primary and secondary education. At a university level, studies in this line are scarce. One example is the work by Galindo-Illanes and Breda (2022) related to the notion of derivative in the context of engineering. Sol and Larios-Osorio (2021) carry out the only research focused on the notion of finite limit of a function at a point, without specifying the professional orientation of the considered books.

## METHOD

In order to address our research goal, we conduct a content analysis of textbooks (Rezat & Sträßer, 2015) in which we adopt a qualitative perspective, with a descriptive and exploratory approach.

Regarding the selection of our textbooks sample, we first determined which Spanish public universities offer the considered degrees. Then, for each degree and university, we searched for the academic year 2024-2025 teaching guide of the subject in which multivariable calculus is introduced for the first time. This led to the consideration of 27 teaching guides for the mathematics degree, 29 for the computer engineering degree, and 35 for the economics degree. Then, the bibliography was reviewed for each teaching guide, and all the recommended manuals were listed and counted.

For this paper, we decided to analyze the most frequently recommended textbook in each degree that covers the topic of multivariable functions. Namely:

- Mathematics degree: “Mathematical analysis” by Apostol (1981). This book was recommended by 13 out of the 27 teaching guides.

- Computer engineering degree: “Calculus” by Strauss et al. (2002). This book was recommended by 5 out of the 29 teaching guides.
- Economics degree: “Essential mathematics for economic analysis” by Sydsaeter et al. (2021). This book was recommended by 24 out of the 43 teaching guides.

We have analyzed the most recent English editions available for us. The unit of analysis was the section of the book in which multivariable limits were introduced for the first time. The treatment of this topic was analyzed from the perspective of the OSA (Godino et al., 2007), focusing specifically on the primary objects that emerge in the mathematical practice evidenced in the textbooks under consideration: Situations, language, concepts, propositions, procedures, and arguments.

## RESULTS

The situations proposed in the three books are mostly oriented towards the applicability and role of multivariate limits in the resolution of intra-mathematical situations. The three books use verbal and symbolic language, usually presented in a coordinated manner. However, while the books corresponding to the mathematics and engineering degrees also include the use of graphic language, this is not the case for the economics textbook, which does not include the graphic register in relation to the notion of the limit in a multivariable function. Furthermore, we note that the mathematics textbook is predominantly symbolic and verbal, while the engineering textbook is more symbolic and graphic.

From a conceptual point of view, the definition of the limit of a function of several variables is approached in two different ways. In the mathematics book, the definition considers functions between metric spaces, while the engineering and economics books only consider the concept of limit for functions between $\mathbb{R}^n$ and $\mathbb{R}$. These different approaches impact the associated concepts that have to be covered by the books. Only two of the books take procedures into account: the mathematics and the engineering book. The mathematics book uses procedures linked to the definition of limit and propositions previously presented, while the engineering book limits itself to using trajectories as a procedure to arrive at the non-existence of limit in each of the solved problems presented.

Two books contain propositions: the mathematics and the engineering book. We identify two types of propositions. Some of them are related to theoretical aspects and appear only in the mathematics book (sequential criterion, uniqueness of limit, etc.). Other propositions are important from the point of view of computational aspects of the limit. They appear in both books, and they deal with limits and operations or with some kind of criteria for determining the non-existence of a multivariable limit. These propositions play a more central role in the engineering book and, in the case of the mathematics book, their introduction forces the author to abandon his general abstract approach. Finally, two of the books contain arguments, although the mathematics book makes extensive use of argumentations, since it contains many formal proofs. These proofs follow a deductive approach and they are usually based on previously introduced definitions and results, but also on previous knowledge considered to be "elementary" by the author. On the other hand, the engineering book lacks formal proofs, so it only presents some kind of argumentation in exercises related to the non-existence of certain limits, in which the previously introduced criterion is used.

## DISCUSSION

The situations identified in the three textbooks highlight strictly intra-mathematical applications. The mathematics textbook emphasizes the demonstration of propositions and briefly introduces some techniques of limit calculation in an exercise left to the reader. The engineering textbook, besides considering this last situation, also contemplates the study of continuity. The economics textbook presents the concept of limit in a multivariable function, but there is an absence of propositions, procedures and arguments on this notion, which we attribute to the existing focus on the continuity of a multivariable function. The three textbooks vary in the registers of representation, like in the analysis of pre-university textbooks for limits of one variable (Arnal-Palacián et al., 2024). The definition of multivariate limit is addressed, on the one hand, by considering functions defined on any metric space (mathematics textbook) and, on the other hand, for a specific family of functions (engineering and economics textbooks). We identify procedures related to the multivariate limit in mathematics and engineering textbooks. It must be said that, in relation to concepts and problems, unlike Thompson and Wiggins (1990), we found that an increase in the rigor of the definition does not imply a greater number of examples and exercises in the book.

Also, while accessing the sample, we have noticed some remarkable differences between editions of the same textbook in the sections devoted to multivariable limits. This also suggests (Schubring, 2022) that further research on the evolution of the treatment of this topic within the various editions of these textbooks could be interesting. Finally, after these incipient results, we think that it would be worth analyzing the practice of university professors of different degrees, to determine what primary mathematical objects they link to their teaching of the concept of multivariable limit.

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# On doing and undoing in general and applied precalculus-calculus tracks

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*For students in the U.S., the transition from precalculus to calculus is notoriously difficult. We explore the potential causes of this phenomenon through textbook analysis from the perspective of APOS Theory. Specifically, we identify broad procedural themes in general and applied precalculus and calculus textbook examples, and we explore the differences in their process demands. Results show significant differences in the processes required by different precalculus-calculus tracks.*

*Keywords: calculus, precalculus, APOS, textbook analysis, doing and undoing.*

## INTRODUCTION

In the U.S., before students enroll in a calculus course, they must complete a sequence of foundational math classes ending in a course called Precalculus. This course is seen as a cap to the knowledge students will need to know to succeed in the calculus sequence and includes topics such as functions, trigonometry, and even complex numbers. Students in general calculus courses are typically STEM majors, whereas applied calculus is a course typically reserved for non-STEM majors, such as business. As a result, Precalculus courses can vary widely in aims and scope. In this system, many students find the transition from precalculus to calculus difficult, and students entering calculus are often severely underprepared for the reasoning required by university calculus courses (Carlson et al., 2015). Once students enter calculus, it often acts as a gatekeeper course, preventing completion of STEM degrees (Weston et al., 2019). Some reasons this transition is difficult for students include misalignment (Barr et al., 2022) and an absence of foundational reasoning skills (Carlson et al., 2015). Our research is motivated by our institutions' current re-examination of their precalculus-calculus tracks. Furthermore, it contributes to the current conversation in mathematics education research about the impact of introductory prerequisite mathematics course sequences on student attrition and retention (e.g., Carlson, 2015; Weston et al., 2019). In this paper, we seek to understand how differences in general and applied precalculus and calculus textbooks may affect students in various tracks of the precalculus-calculus sequence and the potential impact of these tracks on student retention.

## THEORETICAL FRAMEWORK

As a textbook analysis, this study is situated in the cognitive domain of the potentially implemented curriculum (White & Mesa, 2014). In other words, we seek to understand what potential implications particular textbooks may have on learners in the precalculus-calculus sequence, which we define as a *track*. Since textbooks predominantly convey curriculum to learners in mathematics courses (Fan et al., 2013), we study the expository examples in precalculus and calculus textbooks.

We approach this study from the perspective of Action-Process-Object-Schema (APOS) Theory, which posits that schema development occurs in four nominal stages. Repeated systematic *actions* become *processes*. Once computational processes become conceptualized by the learner, they form *objects*. Learners build *schemas* by discerning relationships between various objects (Cottrill et al., 1996). Tall (1997) refines the space between the process and object layers of APOS Theory by defining a *procept*, or a process-concept. Tall (1997) defines three procepts for calculus: function, derivative, and integral, each with its own “doing” and “undoing” processes, which are assumed to be mutually exclusive.

In this paper, we explore further the notion of doing and undoing in precalculus and calculus by removing Tall’s (1997) assumption of mutual exclusivity to determine if processes exist that simultaneously require doing and undoing and how this affects students in the transition from precalculus to calculus. We answer the research questions: (RQ1) What differences exist between calculus and precalculus textbook examples’ “doing” and/or “undoing”? and (RQ2) What types of “doing” and/or “undoing” are there in calculus and precalculus?

## METHODOLOGY

We designed this study as a 2x2 matrix of general and applied calculus and precalculus textbooks (Stewart, 2016; Sullivan, 2018; Hughes-Hallett, 2014; Denette & Kaskosz, 2020), as shown in Figure 1. We considered general and applied textbooks to simulate various possible tracks students in the U.S. can take through the precalculus-calculus sequence, as illustrated by the arrows in Figure 1.

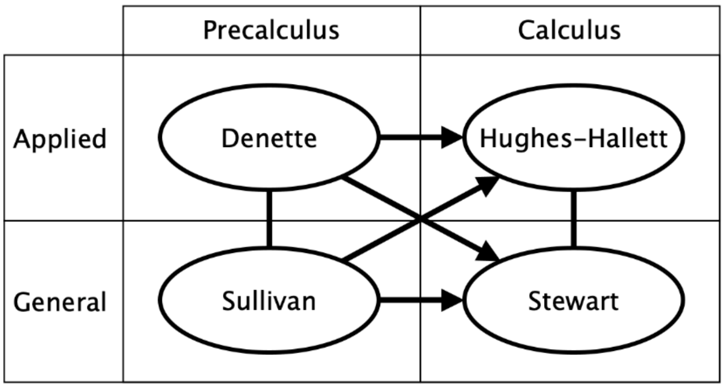


**Figure 1: Study Design.**

From now on, we abbreviate these books as APC, AC, GPC, and GC. For this study, we analyzed examples in the exposition of textbook sections using generative coding (Saldaña, 2013). We included all the sections in APC and GPC (except the Limits chapter in GPC) to simulate a student entering calculus with maximum prerequisite knowledge. From GC and AC, we included corresponding sections to simulate a common calculus curriculum taught using the two texts. A table containing this exact list can be found at https://tinyurl.com/doingandundoing. We collected the coding data, each analyzing one precalculus book and one calculus book as listed in Figure 1, meeting twice to refine the codebook. To answer RQ1, we determined the relative frequency of each code and performed pairwise comparisons described by the edges in Figure 1. To answer RQ2, we identified processes appearing in the examples, which we then classified as “doing”, “undoing”, “both”, or “neither”.

## RESULTS

### Quantitative Analysis

The relative frequencies for examples in the four textbooks illustrating doing, undoing, both, or neither are listed in Table 1.

| | Doing | Both | Undoing | Neither |
|---|---|---|---|---|
| Sullivan (GPC) | 0.395 | 0.091 | 0.231 | 0.283 |
| Denette (APC) | 0.432 | 0.046 | 0.331 | 0.191 |
| Stewart (GC) | 0.350 | 0.420 | 0.107 | 0.123 |
| Hughes-Hallett (AC) | 0.507 | 0.128 | 0.139 | 0.225 |

**Table 1: Relative frequencies of codes in calculus and precalculus texts.**

GPC and APC showed relatively fewer examples utilizing both doing and undoing than both AC and GC. Additionally, APC and AC had lower relative frequencies of examples demonstrating both doing and undoing than GPC and GC, respectively. APC and AC did have higher relative frequencies for examples utilizing either doing or undoing processes than GPC and GC, respectively. AC had the highest relative frequency of examples demonstrating a doing process, although all four showed more examples of the doing processes than the undoing processes. GC has more examples utilizing doing and undoing processes than examples showing just doing or just undoing. GPC has the highest relative frequency of examples using neither doing nor undoing processes.

### Qualitative Analysis

From the analysis of the textbook processes, the following themes emerged. (1) *Evaluating* (doing) is the process of substituting values of a variable to arrive at an answer. *Solving* (undoing) is the process of finding the value of a variable. Evaluating a function at value(s) of its variable(s) is an example of the evaluation process. Solving an equation for $x$ is an example of the solving process. (2) *Simplifying* (doing) is the process of exchanging one algebraic expression for another so that the number of objects in the expression decreases. For example, replacing $x^3x^4$ with $x^7$. *Expanding* (undoing) is the process of exchanging one algebraic expression for another so that the number of objects in the expression increases. For example, replacing $x^7$ with $x^3x^4$. (3) *Composition* (doing) is the process of combining a set of mathematical objects through layering. *Decomposition* (undoing) is the process of breaking down an object into its constituent layers. A classic example of composition is composing a set of functions, while figuring out the functions in a composition is an example of decomposition. (4) *Forward* (doing) and *backward conversion* (undoing) are the transformation of one mathematical object into another such that one direction is objectively more difficult than the other. Conversion between rectangular and polar coordinate systems is one such example. Converting from polar to rectangular coordinates is forward conversion because $x$ and $y$ are presented as formulas in terms of $r$ and $\theta$, which only require evaluation. Converting from rectangular to polar is more difficult, requiring solving inverse trigonometric functions, making it a backward conversion. (5) *Strategize and execute* (both) is a process that requires doing and undoing simultaneously. For example, when evaluating integrals using substitution, the reader must first make the substitution, which requires decomposing the integrand. Then, the integral is evaluated as a "doing" process using a table of antiderivatives. Illustrations of each process from the textbook examples we analyzed are available at https://tinyurl.com/doingandundoing.

## DISCUSSION

The results of the quantitative analysis showed that textbooks differ in the frequency of the processes they ask students to perform. The jump from low relative frequency of examples utilizing both doing and undoing in the precalculus texts to higher relative frequencies in the calculus texts may explain some of the difficulty students experience when transitioning from precalculus to calculus (Carlson et al., 2015). This increase process demand illustrates students are learning not just new concepts, but different problem-solving skills. Notably, this trend is present in the applied and general sequences. It is also noteworthy that the relative frequencies of examples utilizing both doing and undoing processes from APC to GC jumps from 4.6% to 42.0%. This may provide additional barriers for students switching into STEM majors. The jump from GPC to APC is much less severe, 9.1% to 12.8%. Misalignment is one of the reasons for STEM retention issues in the U.S. (Weston et al., 2019), and our quantitative analysis demonstrates a significant misalignment in process demands of precalculus and calculus students.

The results of the qualitative analysis demonstrated that the notions of "doing" and "undoing" as described in Tall (1997) are not mutually exclusive. Some examples in precalculus texts, and even more so in calculus texts require students to *strategize* (undoing) and *execute* (doing) simultaneously. We saw this in examples on evaluating derivatives (e.g., Stewart 3.6 Ex. 3) and integrals (e.g., Stewart 5.4 Ex. 3). Since these topics' examples contained combinations of methods from earlier sections, before attempting a problem, the reader must first discern the combination of methods will get them to a solution. Furthermore, we found that some processes can be classified differently depending on the textbook author's solution. For example, Tall (1997) classifies antidifferentiation as "undoing" because mathematically, it is the inverse operation of differentiation. However, AC and GC present antidifferentiation as a "doing" process, instructing readers to reference an antiderivative table in its solutions. No attention was paid to the inverse relationship between differentiation and antidifferentiation in these examples.

## CONCLUSION

In this paper, we found a large increase in examples requiring simultaneous doing and undoing from precalculus to calculus. This finding may help explain in part why students have such difficulty transitioning from precalculus to calculus (Carlson et al., 2015). Furthermore, we investigated the notions of "doing" and "undoing" processes in calculus and precalculus textbook examples. This study revealed the various processes present in such examples and additionally identified a process that requires simultaneous doing and undoing. This expands upon Tall's (1997) notion of doing and undoing, demonstrating that these two categories are not mutually exclusive in textbook examples. These results illuminate two misalignments, but the impact of these misalignments on students and how they can be minimized remains to be seen.

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# On engineering students' challenges with the delta function

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*Impulsive phenomena are encountered by engineering students in their second or third year of university study. Students must deploy the mathematical model of the phenomena, the delta function also called impulse function, even when they have not had a foundational mathematical definition of this strange mathematical object that is not a proper function. This literature review is a first step of an ongoing design-based-research that aims to connect students mathematical and physical representations of impulsive phenomena. We use Niss et al. (2016) frame of mathematical modelling activities to interpret the existing literature on the mathematical education of engineering and physics students regarding the impulsive phenomena and the uses of the delta function within diverse engineering and mathematics fields.*

*Keywords: engineering, impulsive phenomenon, delta function, modelling, units.*

## THEORETICAL CONSIDERATIONS

Understanding impulsive phenomena requires students to engage in mathematical modelling activities. Students must use the delta function to construct the mathematical model, perform calculations, or interpret results. The framework established by Niss (2016) is used to identify, from the existing literature, potential challenges that students may face in mastering impulsive phenomena, the underlying mathematics and their link. Impulsive phenomena occur in a range of engineering disciplines, including quantum mechanics and electrostatics. Subsequent sections provide examples from various fields to highlight the cognitive challenges involved in modelling activities and showcase the wide array of mathematical contexts where the delta function is used as a tool or as an object (Vos and Frejd, 2020) such as differential and integral equations, and probability. Students must hold a dual description of the delta function as both a mathematical object or/and as a mathematical tool for solving impulsive phenomena problems. According to Niss (2016), through the activities of modelling, the students must: 1) map the physical domain into the mathematical domain where the impulse function is treated as an object (mathematization); 2) compute within the mathematical domain where the impulse function is treated as a tool (calculation); and 3) translate the mathematical result back to the physical domain where the impulse function is again treated as an object (interpretation). We deploy these theoretical tools to explore activities that engineering students encounter when learning about impulse phenomena.

## IMPULSE FUNCTIONS MODEL CONCENTRATED PHENOMENA

When driving through a stop sign, everyone should come to a complete stop. Not everyone does, some people drive though at various speeds as shown in figure 1 (Jackson, 2017). The probability distribution has a spike at $v = 0$; other speeds have much smaller probabilities. These highly concentrated "impulsive" phenomena were analysed informally by engineers and physicists decades before the formal mathematical description (Schwartz, 1981).

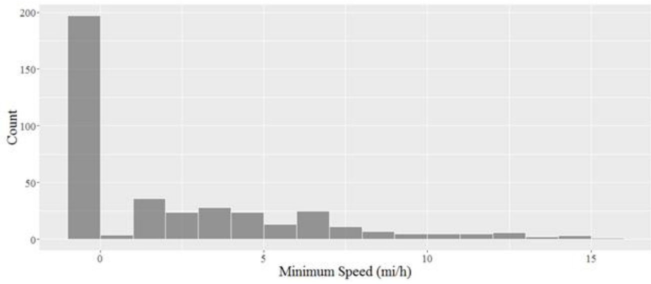


**Figure 1: Histogram of minimum speed through stop sign intersection (Jackson, 2017)**

In probability, the delta function $\delta(x)$ represents probability density concentrated at a single point. If a random variable that takes the value $x_0$ with probability 100%, the probability of the random variable being anywhere else is 0, its probability density function (the probability per unit length) can be modelled using the delta function $p(x) = \delta(x-x_0)$ with the expected value given by $E[x] = \int_{-\infty}^{+\infty} xp(x)dx = \int_{-\infty}^{+\infty} x\delta(x-x_0)dx$. For this integral to yield the obvious result that the expected value is $x_0$, the delta function is defined such that $\delta(x-x_0) = 0$ for all $x \neq x_0$ and has an "infinite" value at $x=x_0$ and has the defining integral property $\int_{x=-\infty}^{x=+\infty} \delta(x-x_0)dx = 1$. Though it is zero almost everywhere, its integral over the entire real line is nonzero, which is atypical for standard functions. The delta function handles a point mass at $x_0$ in a continuous framework (Khuri, 2004) and aids teaching statistical concepts in communication systems (Lopez-Martin, 2004). With the stop sign, the independent variable x is the velocity v, described by the probability distribution P(v) containing a delta function. In quantum mechanics, a wave function describes a particle's position. A particle perfectly localized at position $x_0$, (the particle's position is known with absolute certainty) has wave function as $\psi(x) = A\delta(x-x_0)$. The wave function is called probability amplitude which means that it is not the probability of finding the particle somewhere; it is an indicator that must be transformed to obtain probabilities. The probability density is given by $P(x) = |\Psi(x)|^2 = |A|^2\delta(x-x_0)^2$ with the expected value $E[x] = x_0$. This density is concentrated entirely at $x_0$ indicating that the probability of finding the particle is 100% at $x_0$ and 0% elsewhere. Regardless of how small $\varepsilon$ is (as long as it is non-zero), the probability of finding the particle within the interval centered around $x_0$ is $\int_{x_0-\varepsilon}^{x_0+\varepsilon} P(x)dx = 1$. According to Tu (2023), students of quantum mechanics struggle to mathematize a particle that is confined to a single location into probability distribution concentrated at a single point $\psi(x) = A\delta(x-x_0)$, by incorrectly setting the wave function to a continuous function $\psi(x) = e^{ikx}$ or to a delta at the origin $\psi(x) = A\delta(x)$. They incorrectly calculate energies using integrals containing delta functions (Tu, 2023), such as always evaluating the any delta integral to 1, i.e. $\int_{-\infty}^{\infty} x^2\delta(x-3)dx$ is set to 1 instead of the proper value of $3^2 = 9$. Students have analogous difficulties in electrostatics (Wilcox 2015), mechanical dynamics (Newberry 2008), and mechanics of materials (Jong 2006).

## IMPULSIVE PHENOMENA REQUIRE DIFFERENTIAL EQUATIONS CONTAINING DELTA FUNCTIONS

In quantum mechanics, the Schrödinger equation links the particle's spatial wave function to its energy. The potential energy due to position or configuration relative to other particles is a very narrow and very strongly attracting well. Here, the potential well is the impulsive phenomenon modelled mathematically as a delta function V(x) = $-\gamma\delta(x)$, instead of the position in the previous case. In the Schrödinger equation contains a delta $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)-\gamma\delta(x)\psi(x)=E\psi(x)$. The potential $(-\gamma\delta(x))$ and solution $(\psi(x))$ are shown in Figure 2.

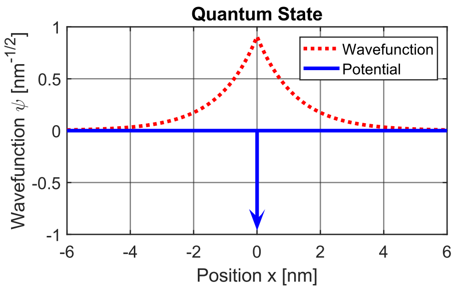


**Figure 2 Graph of the Dirac potential $-\gamma\delta(x)$ and its bound quantum state**

In the calculation step, the impulse function may come at the beginning of the calculations, where an impulse function *input* is integrated or differentiated, such as in the above example. The delta function may emerge as an *output* of the calculation step, such as when calculating the kinetic energy of a quantum state (Gangopadhyaya, 2018) by integrating the second derivative of a known wave function. Students incorrectly perform calculations that involve impulse functions as outputs such as the charge density in electromagnetism (Wilcox 2015). For example, $\int_{-\infty}^{+\infty} x^2\delta(x-4)dx$ is set to $\frac{1}{3}4^3$ instead of the proper value of $4^2$. Students face difficulty in the interpretation step, such as incorrectly interpreting the units of the impulse. If the argument is position x, the units of $\delta(x)$ are inverse meters. If the argument is time t, the units of $\delta(t)$ are inverse seconds. Students claim the impulse is "just a mathematical thing" (Wilcox, 2015; Tu, 2023) and incorrectly assert that it has no units like $\sin(x)$. Unlike $x^2$ or $1/x$ the units of $\delta(x)$ are not obvious from the algebraic expression. Students faces tension between interpreting delta functions as mathematical objects (generally unitless) and as physical objects (which tend to have units).

## DISCUSSION

Engineering students face significant challenges with mathematical modeling, solving the model, and interpreting the results in physical context (O'Brien, 2014; Sullivan et al., 2018; Zandieh, 2000). Difficulties from a gap from theoretical knowledge to practical application and the abstract nature of mathematical models (Thompson, 2014). The delta function is not typically introduced in terms of familiar mathematical notions and students may lack familiarity with its distributional nature, requiring a deeper understanding of generalized functions (Hörmander, 2015). Scholars debate how the delta function should be introduced in engineering courses. Formal, rigorous mathematical models of the impulse include distributions (Schwartz 1981) and nonstandard analysis (Benham, 2014). A formal mathematical introduction could provide a solid foundation for rigorously understanding its properties (Gelfand & Vilenkin, 1964) but risks overwhelming students with unnecessary abstraction. Educators debate how much theory to teach to engineering and physics students (Amaku, 2021). Engineers find distributions difficult (Juric, 2022; Gallardo, 2014) and engineering students reject distributions even after explicit instruction and use naïve forms of nonstandard analysis (infinitesimals) in their explanations (Cavallero, 2004), as the original thinkers (Laugwitz, 1992). In general, researchers agree on the efficacy of introducing a mathematical object through its practical applications before delving into its mathematical definition (Bressoud, 2015). But delving into the definition of the delta function requires nonstandard mathematical notions (distributions) that may be too costly to teach and unnecessary in an engineering context. More research should investigate engineering students' difficulties and conceptions of delta functions as mathematical objects and models.

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# On Narratives of the Definite Integral in Biocalculus

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*While calculus textbooks generally cover the same content, the ways in which the ideas are sequenced and emphasized can greatly impact how students come to understand the content and how the teacher structures instruction. As cumulative changes over time is vital in understanding biological processes, I leveraged the literary tradition of narrative to explore how writers from a variety of disciplinary backgrounds present the definite integral specifically to life science majors.*

*Keywords: disciplinary perspectives, definite integral, narrative.*

Researchers have clearly established the importance of the textbook in the teaching and learning of mathematics; textbooks influence instructors' pedagogical decisions and structure the sequence of key ideas for students (Fan et al., 2013). In analysing how 5 calculus textbooks answer the question "What is the definite integral?", Huffman Hayes (2024) found that each presented a unique story and invited students to make a variety of potential connections, particularly with physics. Because many colleges and universities also offer a discipline-specific calculus course, with core calculus concepts motivated by and contextualized within life sciences (e.g. see Luque et. al., 2022), I expand on Huffman Hayes' findings. My research question is how does the presentation of the definite integral vary in biocalculus textbooks written by authorship teams with distinct disciplinary-orientations?

## CORPUS

My biocalculus corpus is comprised of 4 textbooks designed for biology-specific Calculus. These texts were chosen from diverse publishers and because the authorship teams represent diverse research areas. To provide counterpoint, I also analysed a widely-used, standard calculus book written by a mathematician:

- **A Mathematician's Calculus:** J. Stewart. (2016). *Calculus: Early transcendentals* (8th Ed.). Cengage Learning.
- **Mathematicians' Biocalculus:** J. Stewart & T. Day. (2015). *Biocalculus: Calculus for Life Sciences.* Cengage Learning.
- **Biomathematicians' Biocalculus (Example 1):** C. Neuhauser & M. L. Roper. (2018). *Calculus for biology and medicine*. Upper Saddle River: Pearson.
- **Biomathematicians' Biocalculus (Example 2):** E.N. Bodine, S. Lenhart, & L.J. Gross. (2014). *Mathematics for the life sciences*. Princeton Univ. Press.
- **Biologists' Biocalculus**: A. Garfinkel, J. Shevtsov, & Y. Guo. (2017). *Modeling life: the mathematics of biological systems*. Springer.

For me, *biomathematician* refers to a mathematician whose research is directly related to biology. For potentially ambiguous text classifications given the authors' backgrounds, I used any preface materials that explained the motivation for and process of developing the textbook.

## METHODS

Eye tracking studies of undergraduate STEM students suggest that novices read challenging new texts front to back, unlike experts who tend to “jump around” and return to previous passages (Gold et al., 2021). Accordingly, the sequencing of ideas and distance between connected concepts in a textbook has greater implications for the novice learner than the instructor, who is already a content expert. To map how novice readers are likely to engage with the corpus, I frame my analysis through the literary tradition of narrative, which has been used to study mathematics curriculum (e.g. Dietiker, 2013; Dietiker, 2015; Miežys, 2023; Huffman Hayes, 2024). I conceptualize narrative following Netz (2005), who argues that mathematics texts tell a story, distinguishing between *description* and *narrative*, writing, “some passages - descriptive - add detail to the fictional world, constructing its underpinning of reality; other passages - narrative - unfold the plot that takes place in that fictional world” (2005, p. 262). The final product of my analysis is a set of chronological diagrams that position the mathematical topics (descriptions) along a story arc.

## RESULTS

Figures 1-5 are the resulting narrative diagrams. To interpret these diagrams, I have three general features: upward slope (rising action) indicates a concept building on another concept, horizontal lines (a climax or false-climax) are concepts that were separated from the general text as important (e.g. by a box.), and downward slope (denouement) indicates a closing thought. The majority of the concepts are standard calculus topics. The context-specific problems provided as motivation are the “drug problem” (finding the cumulative amount of a drug delivered to a patient given the rate of drug delivery), the “area problem” (finding the area under a curve), the “distance problem” (using the velocity of a car to find the distance travelled), and the “pathogenesis problem” (area under measles pathogenesis curve gives amount of infection required for symptoms to develop). There are two notable combinations of general features. A horizontal line that ends without a subsequent decent indicates a split narrative, where the reader is to keep this sequence of ideas in mind, but we start again at height zero to build up another sequence of concepts. A horizontal line appearing at a height greater than zero without an upward sloping line preceding it indicates a *deus ex machina*, or the abrupt introduction of a concept in a climactic role without narrative connection to proceeding concepts.

In the Mathematician’s Biocalculus text, the fundamental theorem of calculus (FTC) Part 2 is called the "evaluation" theorem when first introduced. At the stage where differentiation and integration are described as reverse processes, the evaluation theorem is rebranded as FTC part 2. In the Biomathematician’s Biocalculus text (Example 1), the “distance problem" is briefly discussed in antiderviatives, but not emphasized. Within the chapter on integration, the "distance problem" is given as an application of integration, rather than used as motivation. In the Biomathematician's Biocalculus text (Example 2), the "distance problem" appears as an end of section exercise.

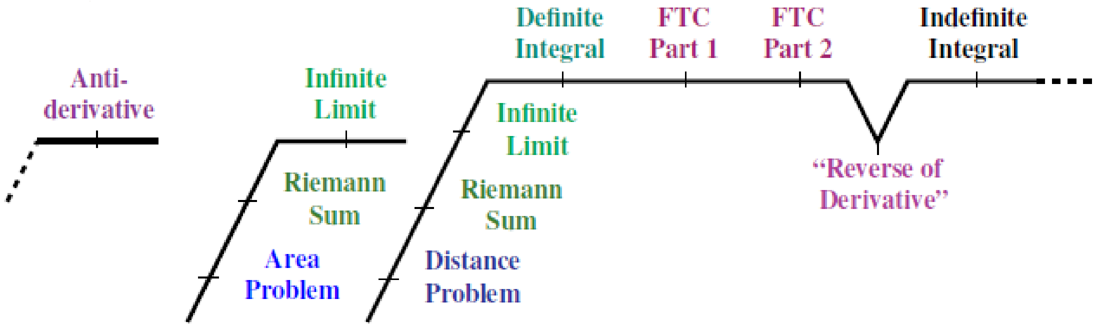


**Figure 1: Narrative diagram for A Mathematician's Calculus.**

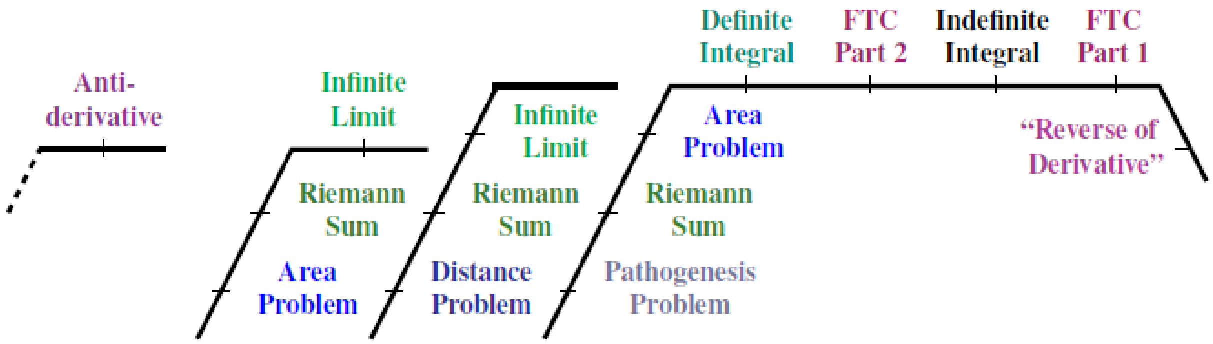


**Figure 2: Narrative diagram for Mathematician's Biocalculus.**

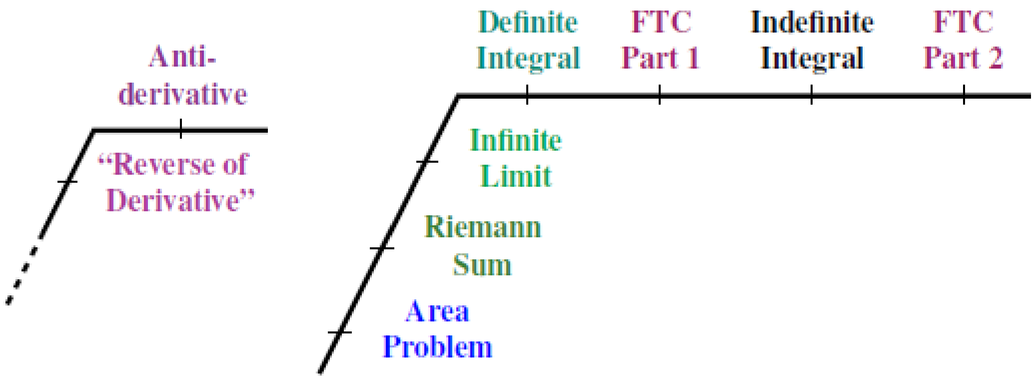


**Figure 3: Narrative diagram for Biomathematicians' Biocalculus Text (Example 1).**

In the Biologists' Biocalculus text, the fundamental theorem of calculus part 2 is simply called the fundamental theorem of calculus; there is no discussion of the fundamental theorem of calculus part 1. Additionally, the fundamental theorem of calculus is first presented in a manner that a mathematician would likely consider an abuse of notation; the same variable is used for the variable of integration and the upper bound of integration. Another feature that makes this exposition distinct from the other texts is the infinite limit is instead expressed as a limit of $\Delta t$ approaching 0 where t is the independent variable of time in the distance or drug problem.

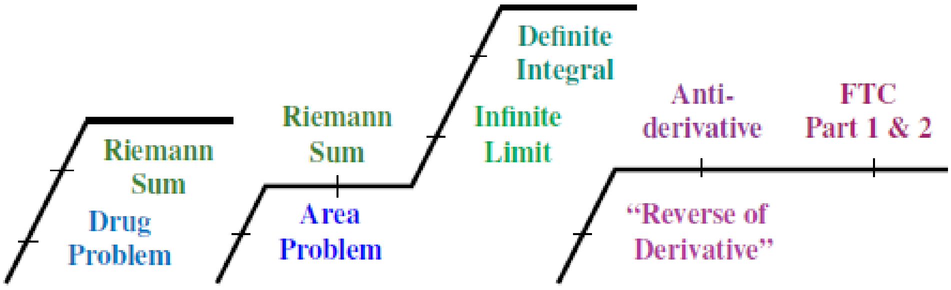


**Figure 4: Narrative diagram for Biomathematicians' Biocalculus Text (Example 2).**

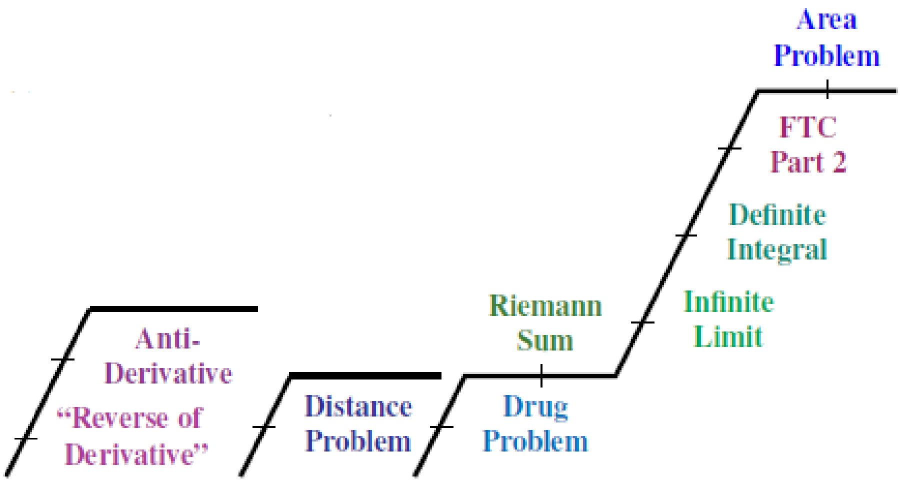


**Figure 5: Biologists' Biocalculus Text**

Each narrative connects a motivating problem (either area, distance, or drug) to the definite integral, however there is a wide variety in the "length" of these arcs, with the shortest being 3 topics and the longest being 10 topics – although in this maximal case (figure 2) the text revisits the area problem just prior to the definite integral.

## DISCUSSION

The narrative diagrams reveal great variation between texts with distinct disciplinary-orientations. In comparing the 5 figures, there are three key findings: (1) while the *deus ex machina* is a popular narrative device in stories of the derivative, e.g., to introduce the derivative function in Biomathematicians' Biocalculus (Example 1), none of these narratives used a deus ex machina to reach the definite integral, (2) surprisingly, in some narratives, the definite integral was not a climax, and (3) there is a wide variety in the arc lengths connecting a motivating problem to the definite integral. To improve outcomes for students, future work should explore the scope and rigidity of students' structural expectations with respect to narrative and how this impacts learning. Certain texts incorporated several split narratives, however split narrative is only common in certain literary genres, such as mystery or fantasy. Are students that encounter split narrative structures regularly more capable of parsing split narrative in mathematical texts? Additionally, curricular materials differ substantially with respect to the cognitive load they place on learners (Sweller et al., 1998); cognitive load theory suggests reading a textbook will place high demand on students’ short-term memory. Because learners tend to read texts from front to back, narrative arc length will have a direct impact on a learner’s ability to synthesize text. What is the optimal arc length? Too short and the ideas may be insufficiently connected for student meaning making, but too long and some students may not be able to make those connections. Finally, in comparison to the narratives explored by Huffman Hayes, which primarily made disciplinary connections to physics, the Biocalculus texts I analyzed drew connections to life science inspired contexts (some to a greater extent than others). Because motivation can improve student engagement (Akbuga & Havan, 2022), future work should explore how the integration of these examples into definite integral narratives impact student performance outcomes.

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# On students' perception of explorative, translation, and example-generation tasks for understanding Calculus in one variable

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*Automatic assessment is widespread in Mathematics, and it integrates computing environments and grading for the creation of meaningful and open-ended tasks for inquiring students' understanding. This study focuses on three types of tasks related to univariate Calculus: explorative tasks, translation tasks, and example-generation tasks. They have been experimented with in two first-year university Mathematics modules in Italy and Sweden as group formative activities. The data consists of students' responses to a survey that highlights their perception of the difficulty of the tasks and of the increase in understanding triggered by the activities. Our results show the difficulty perceived as reasonable and the increase as tangible.*

*Keywords: computer-aided assessment, example-generation tasks, explorative tasks, mathematics education, translation tasks.*

## INTRODUCTION

The impact of automatic assessment tools on higher education has undergone a strong boost in recent years, driven by the need for support for innovative learning approaches. These tools, paired with suitable methodologies, improve the students' experience, providing immediate feedback and enhancing engagement (Barana et al., 2021). Post-secondary Mathematics, including Calculus, is well-suited to this purpose, allowing for the creation of tasks to assess how deeply students understand its topics, thanks to computer-aided assessment (CAA) systems, which combine the potential of advanced computing environments with solid grading capabilities.

This paper presents a study framed within a joint project between an Italian and a Swedish university, devoted to CAA in post-secondary Mathematics (Fahlgren et al., 2024). Specifically, the research focuses on tasks involving univariate functions. Students were administered a survey after completing the tasks to investigate their perception of the tasks' difficulty and the extent to which they felt their understanding had increased. This study helps determine the actual possibility of designing activities that learners find comfortable, thereby providing benefits from both an instructional and a more personal perspective.

## THREE DIFFERENT TYPES OF TASKS

The research is based on three types of tasks concerning functions. In the first type, 'explorative tasks', students are encouraged to use interactive dynamic technology, for example, GeoGebra or Maple, to explore mathematical relationships and formulate conclusions. Suitable use of these technologies is widely recognized for enhancing students' conceptual understanding. In addition, researchers emphasize the value of asking students to formulate their conclusions in writing (Joubert, 2017). The second type of task, 'translation tasks', concerns tasks where students are asked to convert between registers, for example, by determining a function formula for a given graph. It is widely acknowledged that a fundamental aspect of understanding the concept of functions is the ability to transition between various representations, such as formula and graph. Translating from graph to formula is a more challenging task for students compared to the vice versa (Leinhardt et al., 1990). In the third task type, 'example-generation tasks', students are prompted to create examples that meet specific conditions. Since there are no general solving methods for such tasks, students need to be creative and develop strategies based on their conceptual understanding. Moreover, if students are asked to provide more than one example, they have to reflect on how their first example can be varied while still meeting the given conditions. Indeed, the ability to generate diverse and valid examples is indicative of their deep understanding (Watson & Mason, 2005). It has been important to construct these tasks at an appropriate level: they had to be challenging to engage the students properly, but not too hard to avoid them quitting the activity.

## COMPUTER-AIDED ASSESSMENT SYSTEMS

CAA systems recently grown significantly in university-level math education, offering algorithms, customizable grading, adaptive assessments, and interactive feedback. These features make them especially effective for formative assessment, enhancing learning through timely, constructive feedback (Barana et al., 2021). CAA systems personalization of grading algorithms facilitates the automatic assessment of example-generation tasks. Since example-generation tasks often allow for many different correct responses, they are time-consuming to correct manually. Hence, CAA systems are suitable for this task type, especially in large study groups at the university level.

Sangwin et al. (2009) propose utilizing computer-generated representations of mathematical objects to provide implicit feedback rather than explicit solutions. Feedback could include the graph of the student's own response before grading, either by leveraging the CAA system's preview capabilities (Barana et al., 2021) or encouraging students to use external tools like Dynamic Mathematics Software (DMS) to graph their solutions. This allows students to verify whether their function meets the given criteria. Such an approach promotes self-assessment by giving students the opportunity to independently identify and correct their errors (Black & Wiliam, 2009).

CAA systems leverage algorithmic capabilities to generate graphs that can be included in tasks, adapting to random parameters to provide unique variations for each attempt and student. These capabilities enable the creation of graphical tasks—mathematics assessment items in which a graphic plays a crucial role in solving the problem (Lowrie et al., 2012). The graphic conveys key information that students must interpret and decode as part of the solution process. Such tasks are often used to promote students' sensemaking (Lowrie et al., 2012). When graphical tasks pertain to calculus, solving them typically involves interpreting the graphical representation of a function, analyzing its features, and converting the information into another form, such as symbolic or numerical representations. This practice supports students in drawing connections and developing abstract reasoning skills (Adu-Gyamfi et al., 2017).

## METHODS

The study focuses on the following research question (RQ): *how do bachelor students perceive the impact of these tasks' typologies in understanding mathematical topics related to univariate Calculus?* In order to respond to this question, data from a survey to first-year university students taking a module in Mathematics directly involving Calculus have been analyzed. The following questions have been considered:

(Q1) What do you think about the difficulty level of the following tasks?

a) Tasks where you should investigate using interactive graphs and then formulate your conclusions in writing
b) Tasks where you should determine the functional formula based on a given graph
c) Tasks where you should provide two examples of functions that fulfil certain conditions

(Q2) To what extent do you agree with the following statements on tasks?

a) Investigating mathematical relations using interactive graphs and formulating conclusions provided an increased understanding
b) Determining the formula of the function based on a given graph provided an increased understanding
c) Providing two examples of functions that fulfil certain conditions provided an increased understanding
d) It was difficult to formulate conclusions in writing
e) The work with the tasks has been at a reasonably challenging level

Our sample is constituted of 119 biotechnology students from the University of Torino, Italy (69 from the academic year 2022-23, 50 from 2023-24) and 293 engineering students from the University of Karlstad, Sweden (154 from 2022-23, 139 from 2023-24). The module in Turin included also linear algebra and probability/statistics, while in Karlstad only basic calculus was covered. According to ethical guidelines, we considered only students who provided explicit consent to have their responses (in aggregate form) used for research purposes. The data, consisting of responses on a Likert scale, vary from 1 (very low/fully disagree) to 5 (very high/fully agree) and are presented through descriptive statistics.

## RESULTS

Starting with question (Q1), around half of the students considered explorative tasks (item a) as being mid-difficult (score 3), both in Italian and Swedish universities. The other students tended to distribute almost equally between considering the tasks as being of low or high difficulty, with a slight perception of less difficulty in Sweden. Translation tasks (item b) had a sharper perception, with most of the students responding either “mid” (score 3) or “high” (score 4) difficulty and higher averages. Regarding example-generating tasks (item c), “mid” and “high” were again the most frequent responses. However, in Italy, example-generating tasks were perceived as more difficult than translation tasks, whereas in Sweden, the opposite was true. Table 1 summarizes the most noteworthy outcomes: for each university and a.y., the left column presents the average, while the right one (bold) considers the responses given by at least 25% of students, in decreasing order of frequency. We underlined them if they were given by at least 50% of students.

| Q1 items | IT 2022-23 | | IT 2023-24 | | SE 2022-23 | | SE 2023-24 | |
|---|---|---|---|---|---|---|---|---|
| a) | 2.99 | **3** | 3.14 | **3** | 2.78 | **3** | 2.86 | **3, 2** |
| b) | 3.33 | **4, 3** | 3.78 | **3, 4, 5** | 3.32 | **3, 4** | 3.30 | **3, 4** |
| c) | 3.57 | **4, 3** | 3.84 | **4, 3** | 3.12 | **3, 4** | 3.06 | **3, 4, 2** |

**Table 1: Ratings about the difficulty level relative to various kinds of tasks**

Considering now question (Q2), as per items a), b), and c), referring to explorative, translation, and example-generation tasks respectively, there was general agreement that they increased students’ understanding. It was particularly strong in the Swedish case, where averages were for all items and a.y. higher than 4. It was also relevant in the Italian case, with all averages above 3.5 and some higher than 4. Item d) highlighted a perception of moderate difficulty in concluding in written form (averages slightly above 3), while item e) showed adequateness of the challenges posed. Table 2 presents the key insights, following the same structure as Table 1.

| Q2 items | IT 2022-23 | | IT 2023-24 | | SE 2022-23 | | SE 2023-24 | |
|---|---|---|---|---|---|---|---|---|
| a) | 4.09 | **4** | 4.02 | **4** | 4.33 | **5, 4** | 4.29 | **5, 4** |
| b) | 4.01 | **4** | 3.70 | **4** | 4.21 | **5, 4** | 4.24 | **5, 4** |
| c) | 3.70 | **4** | 3.52 | **4, 3** | 4.08 | **4, 5** | 4.16 | **5, 4** |
| d) | 3.30 | **4, 3** | 3.04 | **3, 2** | 3.22 | **3, 4** | 3.12 | **3** |
| e) | 3.71 | **4, 3** | 3.74 | **4, 3** | 3.92 | **4** | 4.10 | **4, 5** |

**Table 2: Ratings about agreement with certain statements on tasks**

## DISCUSSION AND CONCLUSIONS

The results presented above allow us to answer the research question (RQ): students generally perceived the impact of the tasks positively on their understanding, recognizing how they helped them improve it. Moreover, their difficulty has been rated as reasonable, thus allowing them to be adequately engaged without losing interest. These findings confirm students’ good perception of learning calculus with interactive tools, agreeing with other studies such as (Bedada & Machaba, 2022). Future research can consider asking and analyzing more generally how activities of this kind impact learning Mathematics in a broader sense by considering different Calculus topics and other mathematical branches.

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# Partial Derivatives in Thermodynamics

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*An epistemological analysis of the fluidity of the definition of dependent and independent variables in thermodynamics leads to a discussion of the implications for the design and interpretation of contour diagrams in this context. An overview of research looking at expert and student calculations of partial derivatives from contour maps appropriate to thermodynamics, as part of the Paradigms in Physics project at Oregon State University, is presented, along with links to curricular materials developed in this project.*

*Keywords: Teaching and learning of specific topics in calculus, teachers' and students' practices related to calculus across disciplines, partial derivatives, contour diagrams, thermodynamics.*

## VARIABLES IN THERMODYNAMICS

Consider a series of experiments involving a gas in a piston such that the pressure $p$ , volume $V$, and temperature $T$ can be changed and/or measured, see Figure 1.

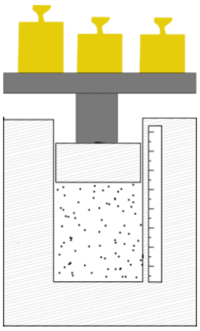

**Figure 1: Gas in a piston. The pressure is controlled by adding weights to the top of the piston. The temperature is held constant by immersing the piston in a constant temperature bath or the entropy is held constant by insulating the piston (not both!).**

In each run of the experiment, the temperature is held constant by immersing the piston in a constant-temperature thermal bath, which allows energy to flow into/out of the piston. The results of these experiments can be plotted as a series of isotherms on a single contour plot, see Figure 2a. Alternatively, in a series of adiabatic experiments in which the piston is insulated so that no energy flows into/out of the piston from the environment, it is the entropy $S$ which is constant, resulting in a different contour plot, see Figure 2b. (Calculating entropy is subtle and often difficult. Holding entropy constant is easy—insulate!)

A Carnot cycle consists of a series of expansions and compressions of the piston, alternating between isothermal and adiabatic, so that the system returns to the same state, see the bold contour in Figure 2c.

In a mathematics class, the very definition of a function implies that the variables $x$ and $y$ are independent and the value of the function $f(x, y)$ is dependent. In contrast, thermodynamics treats all variables equally and which variables are considered to be dependent and independent may change during data taking and analysis.

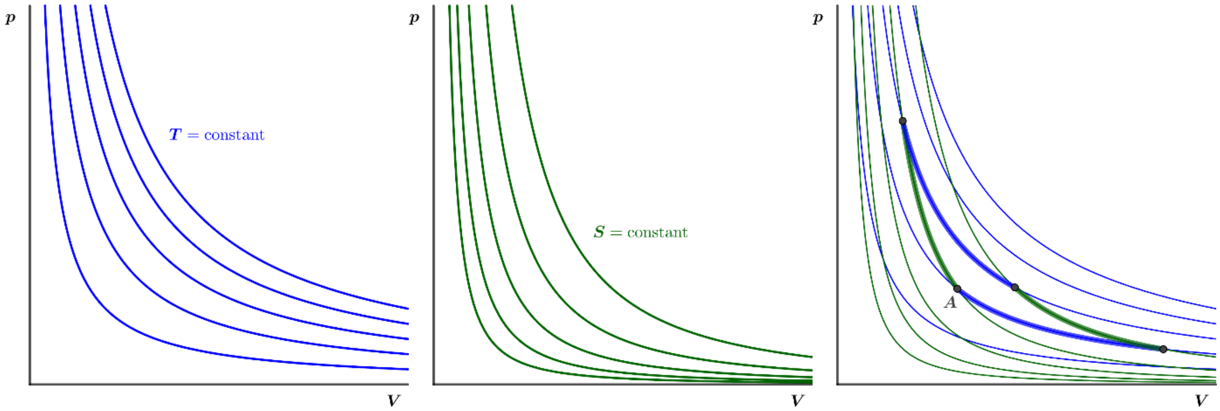


**Figure 2: Contour plots of data from experiments on a piston. Fig 2a shows constant temperature isotherms, Fig 2b shows constant entropy adiabats, and the bold contour in Fig 2c shows a Carnot cycle.**

What does not change is the number of independent variables. For a given experiment, the number of independent variables is equal to the number of ways of getting energy into/out of the system. In the piston experiment, this number is two: through heating and/or through doing mechanical work on the system. A scientist usually thinks of the variables that the experimenter controls as independent. Here, one would consider temperature (or entropy) to be the independent. Then, either pressure or volume can be chosen to be independent, but not both. Experimentally, it may be easiest to control the pressure, by adding or subtracting weights from the top of the piston and then to measure the resulting volume.

## TYPES OF CONTOUR GRAPHS

This flexibility in interpreting which variables are independent and dependent has significant consequences for the interpretation of contour plots. In the conventional interpretation of a contour plot, such as a topographic map of a hill, the independent spatial variables $x$ and $y$ are plotted on the horizontal and vertical axes, respectively, and values of the dependent variable, height, are contours. In Figure 2, from the point of view of the experiment, the independent variables are plotted as contours and on the vertical axis, while the dependent variable, volume, is plotted on the horizontal axis!

Nevertheless, in thermodynamics, it is more common to plot $p$ vs. $V$, as in Figure 2, because in the analysis, most often the scientist wants to calculate the work done on the system given by

$$W = -\int p\ dV,$$

interpreted in standard calculus language as "the area under the curve." Now we have switched our point of view, so that the volume is independent, the pressure is dependent, and the temperature is a parameter that specifies which run of the experiment is being considered.

## DERIVATIVES FROM CONTOUR GRAPHS

Returning to Figure 2c, we might also want to determine the partial derivative $\frac{\partial p}{\partial V}$, the main contributor to the bulk modulus. The figure makes it immediately clear that one must ask, not only at what point, but also along which curve, should I calculate the derivative? Thermodynamics provides a generalization of Leibniz notation for these choices

$$\left(\frac{\partial p}{\partial V}\right)_T \text{ or } \left(\frac{\partial p}{\partial V}\right)_S$$

to indicate the curve with $T$ held constant (and similarly for $S$ held constant).

As an additional complication, in contexts where one does not have an algebraic expression for a function, e.g. discrete tables of data or graphs, finding a derivative (at a point) always involves determining small changes in both the numerator and the denominator variables and then calculating the appropriate ratio. Of course, a ratio of small changes only gives an approximation to the derivative. The limit process to find an exact value for the derivative is impossible. Just as in everyday speech, if you ask, “What is the temperature outside?” the answer “in the mid-20’s (Fahrenheit)” is sufficient to tell you that you need a coat. You wouldn’t bother to specify, “What is an approximation to the temperature outside?” In the same way, scientists will call this ratio “the derivative” so long as the shared understanding is that the approximation is good enough.

What remains is to decide which two points to use to find the ratio of small changes. The two points must lie along a curve for which the desired variable is being held constant—a clear indication that the considerations in the previous section are crucially important. Also, the two points must be close enough together that they lie in the regime where the function is changing linearly, to the degree of accuracy necessary for the application, but not so close that the difference between the values of the numerator (or denominator) variable cannot be sufficiently determined from the accuracy of the information. Here it is crucially important that the scientist have a clear understanding of the accuracy of the data, the accuracy with which the information can be read of the graph, and the size and spread of the fluctuations of the information.

## PHYSICS EDUCATION RESEARCH AND CURRICULAR MATERIALS FROM THE PARADIGMS PROJECT

In the epistemological analysis above, we see that the interpretation of which variables are independent, dependent, or parameters in applied settings like thermodynamics is fluid (pun intended). The interpretation may change between the taking of data and its analysis. Furthermore, the calculation of a (partial) derivative depends on whether the given information is in the form of discrete data or an analytic formula. For more than 25 years, the Paradigms project at Oregon State University has been studying the epistemological differences amongst experts in different fields and the development of these disciplinary understandings in advanced undergraduate students, using a variety of interrelated theoretical perspectives contained within social constructivism. This research work has been intertwined with the development of curricular materials and hands-on manipulatives for thermodynamics and other middle-division physics courses.

A brief review of relevant research from the Paradigms Project can be found in the next section. Relevant curricular materials can be found on our website (Paradigms Team, 2015–2024a), and a short description of those materials that addresses derivatives can be found in Dray et al. (2019). A lovely, short classroom activity that asks students to confront many of the issues raised in this paper can be found in Paradigms Team (2015–2024b).

## RESEARCH ABOUT STUDENT UNDERSTANDING OF DERIVATIVES FROM CONTOUR GRAPHS

Emigh and Manogue (2024) conducted semi-structured, think-aloud, problem-solving interviews with nine Paradigms students asking them to find the derivative at the indicated point $A$ of a $p-V$ diagram, similar to Figure 2c, where the Carnot cycle changes from holding temperature constant to holding entropy constant. Thought of as a graph of a function of one variable, this is a point where the function is continuous, but the derivative is not. We were curious about what reasoning the students might use, from mathematics reasoning ("I can calculate the derivative from the left and the right, but they are different.") to physics reasoning ("Do you mean holding the temperature constant or the entropy?"). Interestingly, the thematic analysis showed that even when these interviewees demonstrated that they understood both math and physics concepts, they usually didn't relate them without prompting. Only one interviewee spontaneously realized that the two derivatives (one for the blue curve and one for the green curve) can be viewed as partial derivatives with the corresponding variables held constant. Sections V and VI of this reference give several detailed transcript examples of student reasoning. Two important insights of this research are: While these learners were initially orienting themselves to the graphs, they were more likely to attend to the labels on the horizontal and vertical axes than the labels on the contours. Interviewees did not use the subscript notation in generalized Leibniz notation for holding a variable constant until they attempted to manipulate some equations symbolically and did not appear to connect this notation to their other understanding of derivatives.

In a separate study, Bajracharya et al. (2019) conducted semi-structured, think-aloud, problem-solving interviews with eight Paradigms students. Interviewees were asked a more difficult prompt: to determine a particular partial derivative from data with some presented in a contour graph and other data presented numerically in a table. To solve this problem successfully, interviewees needed to identify which partial derivative could be found from the data as presented and calculate the correct ratios of small changes. Although this aspect of the problem was not the main focus of the thematic analysis of that paper, section V includes description of the problems interviewees had identifying which variables were present in the table of data and/or the contour graph and in using this information to find appropriate partial derivatives. These tasks are clearly challenging for middle-division students.

## SUMMARY

Variables in thermodynamics like temperature and pressure do not have the same automatic identification as independent or dependent as spatial variables such as $x$ and $y$ do for scalar fields such as electrostatic potential in electromagnetism. This fluidity in the identification carries over to the interpretations of contour diagrams. We have provided some pointers to research from the Paradigms in Physics team and to related curricular materials that may be of use to researchers, curriculum developers, and teachers.

An observation from Emigh & Manogue (2024) states, “As instructors, we were especially encouraged to see that prompts from the interviewer shifted students’ attention dramatically. The prompts consisted of some variant of ‘Did you think about holding anything constant?’ and led most students both to solve the problem and to make sense of it—and they proceeded to use different language (e.g., ‘partial’ derivatives with some variable ‘held constant’) and different notation (e.g., subscripts) than before these prompts.” This observation suggests that small curricular changes may be very powerful.

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# Partitioning Physical Attributes for Riemann Sum Approximations

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*The Adding Up Pieces interpretation of definite integrals (e.g., Jones, 2013) is important for applications in physics and other domains. Research on integration, however, has paid less attention to how students partition physical attributes when constructing Riemann sum approximations for target quantities. We use interview data to demonstrate challenges that calculus-based physics students experienced when partitioning in service of constructing Riemann sum approximations.*

*Keywords: Calculus, Integration, Partitioning.*

## INTRODUCTION

A growing body of research has investigated students' interpretation and construction of integrals, especially when solving problems in physics and other domains (e.g., Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023). One key issue is whether and, if so, how students connect definite integrals with adding up small amounts of a target quantity (e.g., Ely, 2017; Jones, 2013). We interviewed university students in calculus-based physics courses and observed their challenges partitioning physical attributes in ways compatible with adding up pieces in normatively correct Riemann sums. This implies that significant competencies for applying definite integrals to solve problems outside of pure mathematics have been underexamined.

## LITERATURE REVIEW, THEORY, AND RESEARCH QUESTION

Applying definite integrals to solve problems about physical situations requires reasoning in terms of quantities, such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, and much of it has been grounded in Thompson's account of quantitative reasoning (e.g., Thompson, 2011). Most relevant to the present study is work by Jones (e.g., 2013), who found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. Adding Up Pieces consists of (a) partitioning a physical situation into small pieces, (b) approximating the target quantity within each partition piece, and (c) summing small amounts of the target quantity across all of the partition pieces to approximate the total amount.

In further work, Oehrtman and Simmons (2023) proposed that students' attention shifts across three scales when constructing definite integrals. They named these scales the Basic, Local, and Global models. The main situation we will discuss in our results section is based on pressure and force. Using this example, a Basic model relates quantities with constant values (e.g., pressure • area = force, $P \bullet A = F$). A Local model restricts a basic model to small regions where varying quantities are approximately constant (e.g., pressure • "small amount" of area = "small amount" of force, $P \bullet \Delta A = \Delta F$). A Global model is an accumulation based on local models (e.g., the sum of small amounts of force, $\sum P \bullet \Delta A = F$). The present report makes closest contact with one of several points made by Oehrtman and Simmons—that students must decide *how* to partition. In particular, we focused on one task and asked: *How did the calculus-based physics students partition a physical attribute and to what extent was that partitioning compatible with constructing appropriate Riemann sums?*

## METHODS

We interviewed five students enrolled in first semester, calculus-based physics (Spring, 2023) and nine students enrolled in second semester, calculus-based physics (Fall 2023, 2024). The students attended a selective university in the United Sates. Each student participated in a series of three, one-on-one, semi-structured interviews (e.g., Bernard, 1994). Interviews were conducted by the first author (Fall 2024) and by the second author (Spring 2023, Fall 2023). Interviews were spaced a few weeks apart throughout the semester, and each lasted approximately 1 hour. The tasks afforded opportunities to approximate target quantities by Adding Up Pieces, consistent with constructing Riemann sums. Figure 1 shows one task that we adapted from Sealey (2014). Students worked at their own pace and, as a result, some worked on more tasks than others. We videorecorded the interviews and collected all written work.

> Pressure, $P$, applied across a surface area, $A$, creates a total force, $F$. Consider a *vertical* side wall of a tank with a width of 4 feet and a depth of 3 feet (see picture below). Assume that the tank is full of water. The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where $x$ is the depth of the water (the deeper the water, the greater the pressure). Describe a method you could use to approximate the total force from the water pressure on the wall.

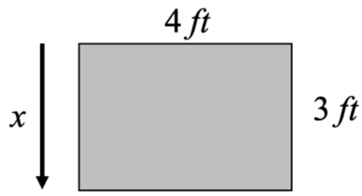



**Figure 1. The Water Pressure task.**

One normatively correct approach to the Water Pressure task is based on the local $15x \bullet 4 \bullet \Delta x = \Delta F$ model which, in turn, is based on partitioning the tank wall into thin, horizontal rectangles that span the wall from left to right (Figure 2 shows a generic example).

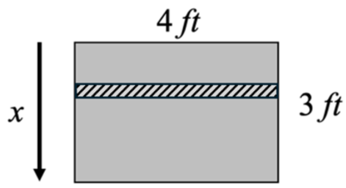



**Figure 2. Normatively correct partition of the tank wall.**

## Results

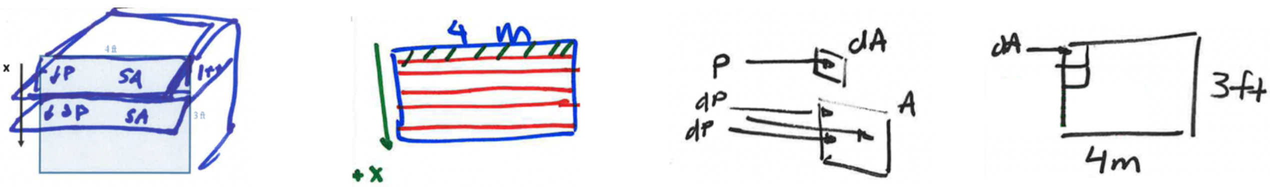
Across tasks, students experienced challenges using rectangular areas to construct Riemann sum approximations. We illustrate this result with the Water Pressure task. For this task, we could observe students' approaches to partitioning when they did not have rehearsed reasoning, because using area to approximate force was less familiar than approximating distance using area under a speed versus time graph. None of the 11 students who worked on the task was confident about the relationship between pressure and force, but each recognized that integration was likely relevant and at some point included either $15x$ or $P$ in an integrand expression. At the same time, none succeeded fully in partitioning the tank wall in service of constructing Riemann sum approximations for the total force. We observed four approaches to partitioning.

### Approach 1: Horizonal Cross Sections Parallel to the Bottom of the Tank

Iliana (Sp 23) discussed Adding Up Pieces and located areas as horizontal cross sections of the tank. She read the Water Pressure task, wrote "$\int 15x$", and explained:

> Iliana: I just said that to calculate the total force you would have to take the integral of the pressure, and then that is like, integral is really just like finding the sum of all the surface areas in the tank. The like, because like the pressure is just applied across each little area for like *x* [gestured with open palms to indicate stacking horizontal slices].

Iliana commented "I don't feel like the 4 feet of the width has anything to do with that, but it might. I am not completely sure." When the interviewer asked where Iliana saw surface area, she drew horizontal slices (Figure 3a) and confirmed that her arrows pointed "downward." Her statements, hand gestures, and drawing gave consistent evidence that she attended to cross sections parallel to the bottom of the tank.




(a) (b) (c)

**Figure 3. Pieces of area. (a) Iliana, (b) Melinda, (c) Matthew.**

### Approach 2: Rectangular Regions Starting from the Top of the Tank Wall (4$x$)

Alex (Sp 23) and Melinda (Fl 23) referenced Adding Up Pieces, and they discussed pressure and area as functions of $x$. As an example, Melinda graphed $P = 15x$, wrote $\int_0^3 15x\,dx$, and explained approximation in terms of adding pressures at different depths: "If we, say, measure at like more points, right? At all these, all these like $x$ values and then we add them all up, then we can get a better approximation of the total pressure that's going against the wall." A few exchanges later, she shaded a rectangular strip at the top of the wall (Figure 3b), wrote $4 \bullet (x)$ to express the area of that region, and explained how to approximate her integral from the graph of $P = 15x$:

*Melinda*: You would want to find like the area of like a little piece, or like one of these, right. And that would be $dx$, which is the change in $x$, times the height, which is 15 times $x$, or like it follows this function. And so, to find the area of that, it's like a rectangle. So, you just multiply the base times the height and then you find the area of that.

Melinda recognized that rectangles on the wall and on her graph of $P = 15x$ referred to different combinations of quantities, but she did not connect local rectangular regions under her graph to local rectangular regions on the tank wall.

In a related approach, Rahul (Fl 24) discussed adding up pressures, wrote the integral $4x \int_0^x 15x'\, dx' = F$, and explained that $4x$ was the area of the entire tank wall.

**Approach 3: Regions That Tile But Do Not Span the Tank Wall**

Matthew (Fl 23) considered two ways to construct a small piece. First he considered pressure applied to little pieces of area and wrote $F = \int P\, dA$. Then he considered little bits of pressure applied over the entire tank wall. He explained why he chose the latter over the former:

Matthew: I don't know if it's going to be area, *dP*, like a little bit of pressure over the entire area, or it's this total pressure over the entire space of the area. But given that the area is constant, and I guess the pressure's changing, I think I would actually not go with [$P\, dA$] and do, maybe *dP* times $A$.

Matthew produced drawings to indicate how he visualized his two approaches (Figure 3c). For the drawing with several arrows labelled "*dP,*" he explained "when you, when you sum up all these *dP*s, you would get the pressure." Figure 3c also shows where on the wall he saw $dA$. Notice that he apparently intended to tile the tank wall, but not with rectangles that spanned the width, in contrast to Figure 2. A few exchanges later he explained that the pressure was constant on horizontal lines that spanned the width of the tank wall, but he did *not* revise his partitioning to align with that in Figure 2.

**Approach 4: Thin Vertical Slices That Span the Tank Wall (4d*x*)**

Four of the remaining five students (Jack, Sp 23; Larisa, Sp 23; Torin, Fl 23; Jeff, Fl 24) located a small piece of area as a thin rectangle that spanned the tank wall, even if they encountered other challenges constructing and interpreting integrals to solve the task. Grace (Fl 24) located a small piece of area similarly, but only after tiling the tank wall using small square regions for $dA$ —similar to Matthew (Figure 3c)—and effortful reasoning about units. Finally, Kathrine (Sp 23) generated $\int 15x\, dx$ but used trapezoids instead of rectangles to determine the area under the graph of $15x$.

**CONCLUSION**

All 11 students who worked on the Water Pressure task considered definite integrals and referenced Adding Up Pieces at some point, but six partitioned the tank wall in ways that were *not* compatible with constructing appropriate Riemann Sum approximations, at least initially. As students move from constructing Riemann sums in one dimensional situations, such as approximating distance from speed in linear motion, issues of how to partition can become more complex. These results suggest that teachers should discuss with students how to partition, especially when the physical situation contains more than one dimension. One possibility is to ask "Where is the integrand constant?" and to use the answer to guide partitioning. In case of the Water Pressure task, such discussion might orient students to the partitioning shown in Figure 2.

**ACKNOWLEDGEMENT**

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# Physical Chemistry Students' Reasoning Related to Spatial Dimensions in the Particle-in-a-Box Quantum Mechanical Model

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*In the undergraduate chemistry curriculum, physical chemistry stands out as one of the most difficult courses given its reliance on calculus, particularly when describing atomic and molecular structure with quantum mechanics. Though there is literature in chemistry education research investigating students' conceptualizations of quantum chemistry, few of these studies focus on students' use of mathematics in this context. To this end, we interviewed students about the "particle-in-a-box", one of the first quantum models discussed in physical chemistry. A preliminary case-study analysis informed by the knowledge-in-pieces framework demonstrates the ways students attend to features related to spatial dimensions might impact their conceptualization and use of quantum mechanical models.*

*Keywords: chemistry, quantum mechanics, calculus*

## INTRODUCTION

Physical chemistry is a uniquely challenging course for chemistry majors. This course is typically taught sequentially over two semesters and covers essential chemical concepts such as kinetics(Bain & Towns, 2016), thermodynamics (Bain et al., 2014), and quantum mechanics (Fox & Roehrig, 2015). Among the three, quantum mechanics stands apart due to its abstract concepts which are difficult for students to conceptualize (Körhasan & Wang, 2016; Partanen, 2018). Generally, students invoke concepts from classical mechanics when interpreting quantum mechanics (Allred & Bretz, 2019), which is similar to the findings reported in physics education journals (Krijtenburg-Lewerissa et al., 2017). Quantum mechanics makes use of mathematical models to describe real-world phenomena, such as atomic structure. There are few studies in chemistry education literature that specifically focus on students' conceptualizations of these quantum mechanical models (Muniz et al., 2018), and though these articles provide a great foundation for further research, many of the studies conducted in chemistry education do not explicitly investigate students' use of mathematics. This gap does not insinuate that quantum mechanics is taught in chemistry without mathematics, but rather that chemistry education research often borrows findings from physics education research to contextualize students' use of quantum mechanics. Indeed, students' difficulties with quantum mechanics is partially due to the mathematics it involves (Fox & Roehrig, 2015), which is unsurprising considering mathematics ability is often indicative of a student's success in chemistry (Derrick & Derrick, 2002). Therefore, we began our study with the aim of investigating how physical chemistry students' reason about spatial dimensions in relation to the particle-in-a-box quantum mechanical model.

## Particle-in-a-Box Model

Briefly, the particle-in-a-box model is a mathematical model used in physical chemistry courses to introduce how translational movement is described by quantum mechanics. There are three variations of this model that describe a particle that is trapped in a one-dimensional, two-dimensional, and three-dimensional theoretical box. With this model, students are taught about the Schrödinger equation, which details a particle's kinetic and potential energy by taking their sum, called the Hamiltonian, and multiplying it by a wavefunction which is an eigenfunction. The Schrödinger equation primarily involves the use partial derivates to describe the particles energy, and during instruction, students walk through the process of normalizing the wavefunction to produce practical equations for laboratory measurements.

## METHODS

For our ongoing study, we were interested in interviewing students currently enrolled in a quantum mechanics physical chemistry course, however, these courses often have low enrollment and are only taught during certain semesters. To ensure we obtain saturation of themes from our sample of physical chemistry students, we decided to reach out to multiple chemistry departments within the University of Wisconsin System, which includes 13 institutions that range in size and level of degrees offered. We are currently in the early stages of data collection, and so far, reaching out to six of these institutions has yielded five participants. Students received $25 gift cards for participating in an hour-long, in-person interview. All involved researchers had completed training on ethics when conducting human-subjects research, IRB approval was provided to conduct this study, and students consented to participate in the project. We conducted semi-structured interviews that consisted of multiple parts: (1) a card-sorting task in which students were prompted to sort equations and representations from the particle-in-a-box model (Figure 1); (2) a problem-solving task in which students were presented with real-world application questions, asking them to calculate the quantum energy of the electrons in a molecule. The molecule used in this context consisted of a chain of carbons (typical application of the particle-in-a-box model) bound to a ring of carbons (atypical).

**Equations and Graphs from Card Sort**

**Equation A**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

**Equation B**

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$$

**Equation C**

$$E_n = \frac{n^2h^2}{8mL^2}$$

**Equation D**

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2}\right) = E\psi(x,y).$$

**Equation E**

$$\psi_{n_x,n_y}(x,y) = \frac{2}{L}\sin\left(\frac{n_x\pi x}{L}\right)\sin\left(\frac{n_y\pi y}{L}\right)$$

**Equation F**

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2)$$

**Equation G**

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi(r)}{dx^2} + \frac{d^2\psi(r)}{dy^2} + \frac{d^2\psi(r)}{dz^2}\right) = E\psi(r)$$

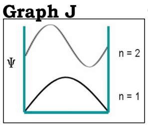
**Equation H**

$$\psi(r) = \sqrt{\frac{8}{V}}\sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)\sin\left(\frac{n_z\pi z}{L_z}\right)$$

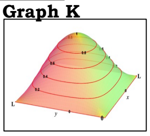
**Equation I**

$$E_{n_x,n_y,n_z} = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

**Graph J**




**Graph K**



**Molecule from Real-World Application**

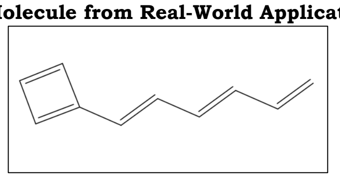


**Figure 1. The equations and representations used for the card sorting activity in the interviews. The labels for the equations were not presented to students but are included here for ease of discussion.**

In this study we used the knowledge-in-pieces (KiP) framework to inform our analysis (diSessa, 1993). KiP describes students' knowledge as a collection of knowledge elements, that are constructed by an individual and when solving problems, these knowledge elements are activated as students make sense of specific tasks. To analyze the data, we processed the data by transcribing and refashioning the transcript, which included added parenthetical descriptions for context (e.g., including an equation's name in place of "this"). These interviews were analysed inductively using narrative coding to generate preliminary themes.(Heisterkamp & Talanquer, 2015) For this conference proceeding, we focus our analysis on two interviews which were collected from the same university, Steve and Alex.

## PRELIMINARY FINDINGS

When presented with the card sorting activity (Figure 1), Steve recognized a couple of the equations from instruction, "At this point…I know [Equation A and B] is the particle-in-a-box, and so one dimensional". Upon recognizing the model as the particle-in-a-box, Steve began by grouping equations A, B, and C. His grouping of the equations by model (e.g., one-dimensional model, two-dimensional model, etc.) stayed consistent for the duration of the task. In comparison, when Alex was presented with the card sorting activity, she recognized, "They're all wave equations for the most part." When she began to create groups, she attended primarily to the variables that the equations included, "So, put energy stuff together for now… [Equations A, C, D, F, I] all have Planck's constant." After sorting the equations into groups, she saved the plots to be sorted last, "[Plot K is] the real question. I have no idea what is going on with [Plot K]." With her attention focused on the plot, she began to look at the axes, "Here's some quantum numbers that probably would be nice to see … [Plot K is] energy in three dimensions. No, it's two dimensions." After seeing the plots' dimensionality, she began to restructure her groups as, "graphical representation sorted by dimension, energy sorted by dimension, and just the plain wave equations sorted by dimensions."

In the second part of the interview, in which students were asked to calculate the energy of the lowest quantum energy level available to a molecule, both Alex and Steve took similar approaches. When Steve began to solve this problem, he recognized, "This is something we [were] talking about in class … this is very similar to the one-dimensional wavefunction for the particle in a box," and chose to apply Equation C. Alex also chose to apply Equation C, but because, "[it's] the one that incorporates all the values I need. This isn't asking about any sort of extra dimensions [like] all the other ones have. It's no integration or derivation involved, so it's very straightforward." Here, both Alex and Steve's initial impressions when grouping the equations continued to influence their problem solving. Though these guiding decisions pointed both of our participants to apply Equation C to solve this problem, their applications differed significantly specifically when deciding on the length parameter. Steve applied

Equation C by excluding the parts of the molecule that varied from the example he had encountered in class, “[that] just logically makes sense to me. The fact that it looks very similar to a problem we’ve seen in class. I honestly don’t know what to do with this [ring]. I’m just going to leave it out.” Whereas Alex included the length of the ring in one dimension, not mentioning the impact this assumption would have on her application. In the end, both students calculated a final value, however, after reflecting, Steve indicated more confidence in his answer in comparison to Alex, who stated “There’s one question answered … it’s just a guess.”

## DISCUSSION AND NEXT STEPS

Though both of our students took similar approaches when solving these problems, the specific features attended to within the context played a significant role in their problem solving. Steve’s more holistic approach that involved focusing on the dimensionality of the equations, highlighting the connection between the dimensionality of the provided molecule and how it related to past problems (excluding the ring in the molecule). On the other hand, Alex initially attended to the variables involved in these equations but was not prompted to consider dimensionality until focusing on the graphs; with dimensionality not being a salient feature, her attention to spatial dimensions was not considered when applying the equations to the molecule. Instead, Alex attended to the amount of calculus that her application would require, choosing the equation that she knew would avoid more advanced mathematics such as integration or differentiation. Importantly, Alex was not the only student in our sample to express these concerns regarding the mathematics. Although not discussed here due to space constraints, our interview protocol also involved asking students questions regarding their perceptions of their preparation for the mathematics in physical chemistry, emphasizing the role of calculus (e.g., *How well do you feel your calculus courses prepared you for the mathematics used in your other chemistry courses?*). As part of this, our ongoing efforts for this project includes not only scaling up with additional data collection, but expanding our analysis to emphasize trends related to students’ experiences with calculus and its use in the chemistry curriculum. Given the role of physical chemistry as the capstone sequence in the chemistry curriculum, we are interested in students’ retrospective views about calculus and its application. Moreover, informed by our framework, we are focusing on the specific knowledge elements students use throughout the interview, including epistemic assumptions about the models employed and fine-grained ideas students associate with the equations and graphs (symbolic and graphical forms), particularly in relation to their interpretation of differential equations (Rodriguez & Jones, 2024; Sherin, 2001). With this work, we aim to highlight the ways students are expected to apply calculus while pointing to the need for further discussion surrounding the ways we can support students in feeling more comfortable with applying complex and abstract mathematical models.

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# Potential and limitations of resources for supporting students using mathematics in physics: a case study about differential equations

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*In the context of science students' high failure rates at the beginning of university, we study resources supporting students facing difficulties which stem from differences between mathematics in mathematics and physics courses. Using the concepts of praxeology and didactic contract, we analyse a resource for students concerning the use of differential equations in physics. We determine what characteristics make it susceptible to support students and the potential and limitations of such resources.*

*Keywords: Teaching and learning of calculus, mathematics for physics, curriculum resources for non-specialist students, praxeology, didactic contract.*

## INTRODUCTION

Difficulties faced by non-specialist students, especially students majoring in experimental sciences, are a key issue in the context of high failure rates at the beginning of university, particularly in France where our study takes place (French Ministry of Higher Education and Research, 2023). It is also increasingly investigated by research (González-Martín et al., 2021). One of the causes of these difficulties is the differences between mathematics in mathematics courses and mathematics as it appears in other science courses, in our case, in physics. Karam et al. (2019, p. 49) showed profound epistemological differences between mathematics in mathematics and physics courses. For example, mathematical work in physics is influenced by the systemic treatment of units: e.g., one cannot add up two quantities of different physical units. One of the researchers' recommendations regarding these differences is to make them explicit to students. However, understanding these differences and bridging the gap between mathematics and physics is often left to the students. Remediating these difficulties remains a significant issue. In this context, we study resources used in France which were designed to make these differences explicit.

## THEORETICAL FRAMEWORK

Within Chevallard's anthropological theory of the didactic (Chaachoua et al., 2019), praxeologies model knowledge through a quadruplet $[T, \tau, \theta, \Theta]$ composed of a type of tasks T (e.g. 'Determine the velocity'), a technique $\tau$ which can be used to solve tasks of that type (e.g. 'Compute the derivative of the position'), a technology $\theta$ which justifies the technique (e.g. 'The derivative of the position function is equal to the velocity'), and a theory $\Theta$ justifying the technology. The technique can itself be composed of one or several types of tasks called ingredients of technique (e.g. 'Compute the derivative of a position' is an ingredient of technique). Types of tasks can be *intrinsic*, meaning they only appear as ingredients of techniques for other types of tasks, or *extrinsic*, meaning they appear as types of tasks themselves.

Brousseau (1997) describes the didactic contract as a set of rules, often implicit, which describe the expected behaviours of the teacher and students *relative to knowledge*. For example, students expect and are expected to use the notion referenced in the title of a section or chapter when solving tasks within that section or chapter. The didactic contracts in mathematics and physics are different, for example, in terms of expected notations (e.g., in France, $f'$ in mathematics and $\frac{df}{dx}$ in physics).

## BACKGROUND

### Difficulties caused by the differences between mathematics & mechanics

In our previous work within physics, but in the specific case of mechanics, we identified three types of differences between mathematics in mathematics and mechanics courses, and we showed that students do face difficulties stemming from these differences (Hellio et al., 2025). First, in mechanics, most types of tasks that incorporate mathematics are intrinsic, meaning they only appear as ingredients of techniques. This leads to difficulties because students need to identify the mathematics tasks that they need to achieve to solve the extrinsic physics task. For example, in an exercise where students need to determine the velocity of an object, the technique can be to "compute the derivative of the position", which incorporates mathematics (Brunel et al., 2015, p. 775) (Difficulty 1). Second, we noted techniques in mechanics which blend mathematics and physics. For example, to solve the type of tasks "determine the position of an object", students must first recognise that there is a differential equation to solve and then solve it (Brunel et al., 2015, p. 774) (Difficulty 2). Third, the didactic contracts in mathematics and physics are different. For example, this causes students trouble in interpreting $\frac{dv}{dt}$ as the derivative of $v$ as a function of $t$ (Hellio et al., 2025, for more details) (Difficulty 3). We consider that if a resource addresses these three difficulties, then it holds the potential to support students in the transition between mathematics in mathematics and physics.

### Presentation of the Maths4Sciences (M4S) resources

The resources we study were created by a group of mathematics and physics teachers led by a researcher in physics education[1]. The group worked between 2017 and 2022 and produced a series of quizzes to test and improve mathematics skills in a physics context. Alongside these quizzes, they designed 56 tutorial sheets on subjects ranging from "Using the Pythagorean theorem" to "Computing the sum of vectors" to provide the methods for solving the problems in the quizzes. The tutorial sheets are generally composed of one to two solved examples, followed by a "Method" section. The examples are in a physics or chemistry context, while the method is in no scientific context. We analyse one of these tutorial sheets named "Solving a first-order

[1] https://maths4sciences.ens-lyon.fr

differential equation by way of identification"[2], which comprises two examples and a section regarding notations at the end. The first example (Ex. 1) of the tutorial sheet is about radioactive nuclei, and the second one (Ex. 2) is about the temperature of an incompressible system. We note that the potential of a resource depends on its use. Here, we analyse the Maths4Sciences sheets assuming students who use them do so on their own and to solve any exercise, not just ones from the designed quizzes.

## RESEARCH QUESTION & METHODS

Our research question is: What is the potential and what are the limitations of an M4S tutorial sheet for helping students overcome the three identified difficulties?

We start by analysing the types of tasks in the tutorial sheet and the associated techniques and technologies. We break the technique down to the ingredients composing it. Then, for each praxeology, we analyse potential didactic contract rules in the form of what students could infer from the context (name of the method sheet), the notations manipulated, and what aspects of the praxeologies remained implicit in the method sheet. We attach particular importance to what is implicit or explicit in the sheet, as making differences explicit may support students (Pospiech et al., 2019).

## RESULTS

The tutorial sheet addresses all three of the types of difficulties. In the next three subsections, for each difficulty, we detail the characteristics of the tutorial sheet addressing it. To get a global view of the tutorial sheet, Figure 1 shows all types of tasks gathered from its analysis. A type of tasks appearing below another means the former is an ingredient of the technique of the latter. A full purple background shows the types of tasks for which there is a technique explicitly described. If there is no explicit technique, the background is shaded.

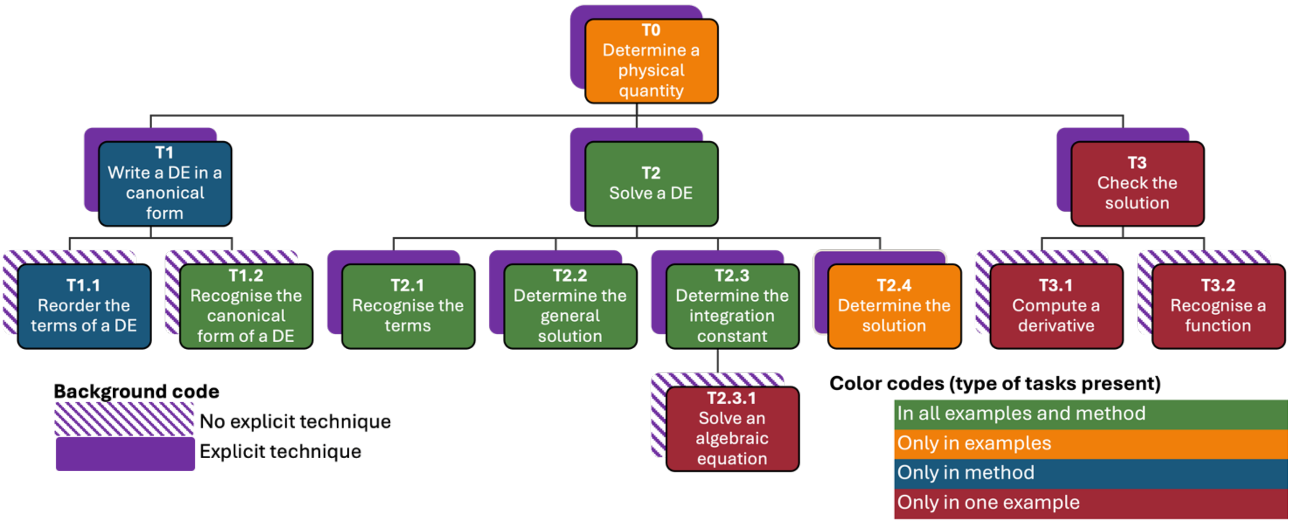


Figure 1: Types of tasks appearing in the tutorial sheet[3]

[2]The author's translation of the tutorial sheet is available at the following link: https://bit.ly/3Z8eYrb.
[3]For high-quality and black and white versions of Figure 1, see https://bit.ly/3Z8eYrb.

### Intrinsic mathematical types of tasks appearing in physics

The extrinsic type of tasks in both examples is a physics one: either "Determine the number of radioactive nuclei" (T0 in Ex. 1) or "Determine the temperature" (T0 in Ex. 2). Mathematics appears as an ingredient of technique, for example, with "Solve a differential equation (DE)" (T2). However, we found no technique or technology showing how to recognise that the technique to determine each of these physical quantities is to solve a DE or why to use that technique. Moreover, we assumed students would have access to all 56 tutorial sheets, and since this sheet is named "Finding the solution of a first-order differential equation", students must know there is a DE to solve to go look for this sheet. Finally, "Solve an algebraic equation" (T2.3.1 in Ex. 2) and "Compute a derivative" (T3.1 in Ex. 1) are, respectively, ingredients of "Determine the integration constant" (T2.3) and "Check the solution" (T3).

### Techniques blending mathematics & physics

We present examples of techniques blending mathematics and physics in the sheet. First, in the case of "Recognize the terms" (T2.1), the Method section says: "Identify, in the physical system studied, the quantities corresponding to $y$, $x$, $a$ and $b$." This blends mathematics notations (see the next section) and the use of the physics system described by the exercise. Second, the technique for the type of tasks "Determine the integration constant" (T2.3) uses the notion of an "initial" (in Ex. 1 and 2) or "particular" (in the Method) condition on the quantity studied, which involves finding this condition within the text of the physics exercise and solving an algebraic equation (T2.3.1 in Ex. 2).

### Different didactic contract rules in mathematics & physics

The difference in didactic contract rules we previously identified between mathematics and physics were differences in notations. Here, the tutorial sheet incorporates "Recognize the terms" (T2.1) as an ingredient of "Solve a DE" (T2). This type of tasks makes the transition more explicit between the physics expression $\frac{dN}{dt} = -\lambda N(t)$ and its notations $N$ and $\frac{d\square}{dt}$ and the mathematical expression a $y' = ay$ and its notations $y$ and $\square'$ (Ex. 1), or between $\frac{dT(t)}{dt} = -\frac{hs}{mC}T(t) + \frac{hs}{mC}T_e$ and $y' = ay + b$ (Ex. 2). Moreover, the final section, called "Caution: Variations in notations", explains there are different common forms for a DE in mathematics and physics and how this changes the way the form of the general solution is expressed.

## CONCLUSION & DISCUSSION

First, the tutorial sheet does not seem to support students in recognising there is a differential equation to solve because the students must already know they need to solve a differential equation to pick this tutorial sheet. We consider this to be a limitation of such resources. Moreover, even within the tutorial sheet, there is no explanation of how to recognise a DE in a physical context. We consider that resources which aim to support students facing Difficulty 1 must have a different structure or, at least, within the sections on each mathematical notion, an explanation of how to recognise the notion in a physical context (Difficulty 1). Second, there are several instances where, through explicit techniques, the blending of mathematics and physics is supported. This more explicit blending of mathematics and physics has the potential to help students facing difficulties with techniques that incorporate both mathematics and physics (Difficulty 2). Third, we find that the tutorial sheet presents the common notations of both mathematics and physics. Following Karam et al. (2019), we argue that this explicit presence of notations from both disciplines may support students having difficulties stemming from the differences in didactic contracts (Difficulty 3).

All in all, we posit there is great potential for resources supporting students specifically by making the differences between mathematics and physics explicit, notably in terms of the diversity in notations they have for shared objects, such as derivatives. We keep in mind that the goal is to make differences explicit without disconnecting the disciplines, which have deep-seated historical and epistemological connections. Moreover, it seems that more attention needs to be given to the recognition of intrinsic mathematical types of tasks in physics.

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# Reexamining calculus practice in introductory physics

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*We examine where and when calculus concepts and skills appear in the first semester of introductory physics. Following similar work in other fields, we adopted the Calculus Concept Framework and used the framework to identify which concepts and skills appear in each section of the first twelve chapters of a standard introductory physics textbook. Results were collated and suggest uneven use of calculus concepts.*

*Keywords: physics, mechanics, calculus, instructional practices, textbooks.*

In the United States, ‘university physics’ is taken by physical science and engineering students and is frequently referred to as ‘calculus-based.’ It has as pre- and co-requisites two or more semesters of calculus. In this paper, we describe a number of data points from our work and that of others that together suggest that the nature of the relationship between calculus and calculus-based physics is in need of renewed examination, and that the shared assumptions of departments, textbook authors, and instructors might not reflect the lived experience of students.

## BACKGROUND

Scholars in Physics Education Research (PER) have examined the use of mathematics in physics courses, for both introductory and more advanced levels, from a variety of perspectives. Redish and Kuo note that mathematicians and physicists use mathematics differently: while physicists load physical meaning onto symbols and equations, mathematicians tend not to (Redish and Kuo 2015).

Biza and colleagues suggested that calculus in the disciplines could be a filter or scaffold: in the former case, calculus is used to determine who is qualified for later coursework; in the latter, it is useful as supporting knowledge to enhance success in later coursework (Biza et al., 2022). Several authors reported that students perceive differences between practices in physics and calculus. Hitier and Gonzalez-Martin (2023) reported that “in mechanics, ready-to-use formulas are provided, so students can solve these problems without using knowledge from derivatives nor thinking in terms of covariation.” Noah-Sella, et al. (2023) described differences in the ways that students perceive related problems in physics and calculus, including the perception that physics problems require ‘more thought’ than calculus.

Several prior studies have investigated student understanding of concepts and skills from calculus in introductory mechanics. For example, researchers have described a relationship between a limited understanding of definite integrals and difficulties in physics, in contexts including kinematics graphs (e.g., Bajracharya et al., 2023). The conceptual understanding of integration needed in physics is often not an outcome of standard calculus instruction (e.g., Orton 1983).

In engineering, faculty assessed the value of calculus "for mathematical maturity more than just the actual calculus" content, because "the way [the engineering course] is taught, you can do it without calculus." (Ferguson 2012, Faulkner, et al. 2019). Faulkner et al. (2019) reported that faculty teaching engineering courses felts that their students have difficulties with using and interpreting mathematical models; choosing and manipulating symbolic and graphical expressions; and using computational tools.

**Integration in physics courses is mismatched.**

Jones and colleagues have shown that most students leave calculus with a conceptualization of integrals as area under the curve (Jones 2015). However, more than half of the integrals in US physics textbooks used an adding-up-pieces conceptualization, and under 10% used area (Piña and Loverude 2019).

**Physics majors report limited use of calculus in introductory physics.**

In a prior study, Loverude (2017) reported on the perception of physics majors in a sophomore-level math methods course of the calculus used in their introductory physics courses. While there was considerable variation, the most common result was not encountering much calculus in introductory physics; one student described introductory physics as 'just barely calculus-based.' Comments suggested a strong focus on computing numerical answers to end-of-chapter textbook problems and that most of the calculus was done by the instructor in lecture rather than by students:

> The calculus that we used was mentioned once, and then the students were expected to just memorize the equations of motion instead of deriving them …
>
> …there were many calculus-based derivations that we weren't in any way required to know or understand. We only just used their results to do plug and chug style problems.

## STUDY OF CALCULUS CONCEPTS AND SKILLS IN MECHANICS

Scholars in several disciplines have posed the question of where exactly calculus is used and what calculus concepts and skills are used in courses for which calculus is pre-requisite. Many prior studies have adopted the Calculus Concept Framework [CCF] (Sofronas 2011). The CCF was developed through interviews with experts and identifies concepts and skills essential to calculus understanding (the concepts and skills are the first two sections shown in Figure 1, respectively). The CCF has been used to used to investigate what concepts and skills from calculus were used in courses in differential equations (Czocher 2013), introductory engineering (Faulkner et al., 2020), and general chemistry (McAfee and Rodriguez, 2024).

In a recent study (Loverude, et al., 2024), we performed a similar analysis for introductory physics, investigating the research question: *What concepts and skills from calculus and related mathematics are encountered by students in introductory mechanics, and when are they encountered?* To limit the project and allow for preliminary discussion of results, we coded sections of a course text for calculus concepts and skills. Focusing on the textbook is only a proxy for what students might realistically encounter in the course, as instructors might increase or decrease the emphasis on calculus in lecture or homework exercises. In subsequent work we will extend this analysis to later topics and seek to characterize instructor practice as well and document what calculus students encounter in class and in homework.

### Methods

Three researchers coded chapters 2 through 12 of a standard physics textbook (Resnick et al., 2014). (Chapter 1 is purely a discussion of units.) We coded for instances of the concepts and skills from the CCF in each section. Codes were negotiated and clarified, and we added several extra categories that reflected elements of mathematical practice, including graphs or figures, the use of function notation, the use of the vector concept and unit vector notation, and explicit attention to units or dimensions. By the end of the first-semester analysis, the agreement between coders was very high. For example, of 243 cells in the Chapter 11 spreadsheet, all three coders independently coded the same result on 236 (97% agreement). Results are shown in Table I.

### Results

The most prevalent calculus concept was the derivative, appearing in 8 of the 11 chapters and in 20 of the 62 sections (just over 30%). Integrals appeared in 5 chapters, and 11/62 sections (just under 20%). In contrast, there was not a single epsilon-delta proof. Limits appeared in only four chapters (and 9 sections) and sequences and series in two sections of a single chapter. Algebraic manipulations appeared in every chapter and 61/62 sections, and trigonometry in 10 of 11 chapters and 27 different sections.

Chapter two (one-dimensional kinematics) was one of the highest in terms of CCF concepts and skills coded, and chapter 4 had the highest prevalence of derivatives, though fewer limits and no integrals. The chapters on Newton's laws (5 and 6), in contrast, did not include a single limit, derivative, or integral. Indeed, after chapter 2, students might not see another integral in the course until chapter 7. While some of the skills and concepts identified in the CCF appeared rarely or not at all, other mathematical practices were more common. Units and vectors were considerably more prevalent than core CCF concepts of limits, derivatives, or integrals.

The sequence of topics is mismatched to the sequence in calculus prerequisites, in terms of the use of calculus skills and concepts. Depending on prerequisite structures, students may be enrolled in first-semester calculus at the same time as introductory mechanics. However, the most calculus-intensive sections are early in mechanics; students first encounter integrals in Chapter 2, so the first or second week of the course, far earlier than integrals in Calculus I. Dot and cross products, Calculus III topics, can appear in first-semester physics.

Function notation (e.g., $v(t)$) is used in beginning sections but then largely vanishes. There is some reason to believe that different usage of function notation is a tension point for students in introductory physics (Loverude 2023).

## DISCUSSION

Despite the perception of calculus as necessary for physics, many sections of the

| **Concepts** | **Total Count (%)** | Ch2 | Ch3 | Ch4 | Ch5 | Ch6 | Ch7 | Ch8 | Ch9 | Ch10 | Ch11 | Ch12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Derivative | 30.65 | 4 | 0 | 6 | 0 | 0 | 1 | 2 | 1 | 3 | 2 | 0 |
| Integral | 19.35 | 3 | 0 | 0 | 0 | 0 | 2 | 1 | 3 | 3 | 0 | 0 |
| Limit | 14.52 | 3 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 |
| Approximation | 8.06 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 1 |
| Sequences/Series | 3.23 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 |
| Riemann Sums | 6.45 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 |
| Parametric and Polar Equations | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Continuity | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Optimization | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Transformation | 1.61 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **Skills** | | | | | | | | | | | | |
| Derivative Computations | 6.45 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Techniques of Integration | 4.84 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Limit Calculations | 3.23 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Sequences/Series | 4.84 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Area / Volume | 14.52 | 1 | 3 | 0 | 0 | 1 | 1 | 0 | 2 | 0 | 0 | 1 |
| Parametric Equations | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Polar Coordinates | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trigonometric Manipulations | 46.77 | 2 | 3 | 3 | 2 | 2 | 4 | 2 | 4 | 1 | 5 | 1 |
| Epsilon Delta Proofs | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Listening/Reading Comprehension | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Manipulating Algebraic Expressions | 98.39 | 5 | 3 | 7 | 3 | 3 | 6 | 5 | 9 | 8 | 9 | 3 |
| Facility with Logarithms and Exponentials | 0.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Facility w/ Definitions & Notation | 93.55 | 2 | 3 | 7 | 3 | 3 | 6 | 5 | 9 | 8 | 9 | 3 |
| **Additional Concepts & Skills** | | | | | | | | | | | | |
| Function notation | 14.52 | 4 | 0 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 |
| Something is a function of something | 100.00 | 6 | 3 | 7 | 3 | 3 | 6 | 5 | 9 | 8 | 9 | 3 |
| Units / dimensions | 98.39 | 5 | 3 | 7 | 3 | 3 | 6 | 5 | 9 | 8 | 9 | 3 |
| Vector concept | 64.52 | 4 | 3 | 5 | 3 | 2 | 3 | 0 | 6 | 3 | 8 | 3 |
| Unit vector notation | 17.74 | 0 | 3 | 1 | 1 | 2 | 2 | 0 | 0 | 0 | 2 | 0 |
| Graphs | 33.87 | 5 | 3 | 7 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| Diagrams | 82.26 | 3 | 3 | 5 | 3 | 3 | 4 | 4 | 8 | 6 | 9 | 3 |

**Table 1: Mathematics concepts and skills coded in introductory mechanics textbook.**

introductory physics textbook do not use calculus skills or concepts, and the presence of calculus is not uniform. Instructors and textbook authors might wish to consider the mismatch between calculus and physics practices and the sequences of topics.

By focusing on mechanics, we have not included the significant calculus (including multivariable and vector-valued functions) in the electricity and magnetism portion of the course, which further stretch prerequisites. This will be a focus of future analysis. The frequency of concepts and skills in the textbooks might also not reflect the experiences of students. We have begun to probe the experiences of instructors and students at our university. We have also expanded our analysis to include other textbooks and other versions of the chosen textbook, to see the extent to which practice might change over time or chosen text.

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# Reflections on a *Master Class* in Teaching Modelling to Life Science Majors

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*Traditional calculus courses often fail to equip life science majors with the skills needed to model complex biological systems. UCLA's Life Sciences 30 (LS30) curriculum bridges this gap by focusing on modeling and simulation. To prepare faculty for teaching LS30, we developed workshops emphasizing pedagogy and team-teaching in Year 1. Recognizing faculty struggles with the modeling-centric content, we redesigned the workshops in Year 2 to prioritize problem-solving and deeper engagement with LS30's mathematical concepts. These adjustments underscore the value of content-focused professional development for faculty for the goal of enhancing mathematics education for life science majors.*

*Keywords: mathematical modelling, faculty professional development, biology*

## THE DISCONNECT BETWEEN CALCULUS AND LIFE SCIENCE MAJORS

In the United States, graduates from life sciences majors outnumber physical science majors and are projected to see the fastest employment growth in decades (Okrent & Burke, 2021). Meanwhile, mathematics is central to understanding biological phenomena (Steen, 2005) but the mathematics most life sciences majors take is usually some traditional version of calculus, which hasn't kept up with the evolving needs of the field (Wilson Sayres et al., 2018). Both traditional calculus or "biocalculus" still focus on algebraic and analytic techniques. But modern life scientists seek to understand and predict complex systems (i.e., modelling physiological and environmental processes) that are driven by feedback loops – modelling and analysis of these systems is not possible with the techniques taught in calculus, because they are nonlinear systems. Life scientists need to be able to use technology, handle large datasets, engage in quantitative reasoning, work with simulations, and analyse and interpret dynamical systems – they need to model (Feser et al., 2013).

This evolving disciplinary landscape raises the question *What mathematics should we teach to life science majors?* To address this content issue, UCLA developed the Life Sciences (LS30) curriculum, including the textbook *Modeling Life* (Garfinkel et al., 2017), simulation labs, and homework tailored to life science themes. As summarized by Bennoun et al. (2023) and Garfinkel et al. (2022), LS30 emphasizes modelling and simulating biological systems, not just as examples to see at the end of a section but as the foundation to learning the mathematics required to explain and predict biological phenomena. Topics like bifurcations, chaos, and oscillations – typically reserved for advanced undergraduate or graduate-level study – are introduced in ways that are accessible to first-year biology majors. Many math faculty do not have the biological or mathematical background to appreciate the contributions of mathematical modelling to biology as a discipline. So, even if introductory mathematics content is reformed for life science majors, few faculty members are prepared or confident enough to teach a modelling first curriculum. This knowledge gap presents a challenge: supporting faculty in confidently teaching advanced concepts like dynamical systems, modelling, and simulations in a way that wouldn't require extensive mathematics prerequisites.

To address this educational challenge, we launched professional development workshops to help mathematics faculty (re)learn the mathematics needed for life sciences and gain insights into biological processes. Participants ranged from experienced researchers in biological modelling to novices to the field. In this paper we share our experiences with these efforts, highlighting successes, challenges, and institutional roadblocks, aiming to inspire conversations on overcoming barriers and equipping faculty to transform mathematics education for life science majors.

## OUR WORKSHOPS

Nearly all students enrolling in first-semester calculus at post-secondary institutions plan to major in life sciences or engineering (Bressoud et al., 2015). However, the typical calculus course favours the latter. Even a "biocalculus" course tends to use few examples from biology, giving more examples from geometry, physics, engineering, or even economics. The reason is that single variable differential and integral calculus cannot solve the kinds of problems that arise in biology. Biology's problems are not about optimizing the area contained in a pen for pheasants given 1000m of fencing. Real quantitative challenges include managing predator populations, explaining genetic switches, and understanding cardiac arrhythmias. UCLA's LS30 course replaces traditional calculus with a modelling-first approach, using nonlinear dynamics to address these biological questions. Students simulate solutions, interpret phase portraits, and adapt Python code, shifting focus from algebraic techniques to practical modelling skills and biological applications.

This shift in content and emphasis entails faculty readiness to teach "change equations", vector fields, Euler's method, classification of equilibrium points, translating between phase portraits and time series, relationships between solution trajectories and initial conditions, limit cycles and attractors, bifurcations, bistable systems, and Hopf bifurcations, while having background knowledge of feedback loops, types of feedback in ecological systems, homoeostasis, oscillation and endocrine control systems. Our workshop's goal was to provide just-in-time professional development for mathematics faculty's content knowledge. We have previously reported on the success of the workshops in meeting the stated objectives of improving faculty content knowledge and confidence for teaching dynamical systems to first year biology majors (Czocher et al., 2024). In this report we summarize the two workshops, and focus on the rationale for the changes we made from one year to the next.

We designed the workshops in concordance with best practices for faculty professional development. Effective professional development includes multiple kinds of sessions: engaging in content-focused investigations, readings and brief lectures, observing cases of instruction, analysing instructional materials, and reflecting on student work (Math and Science Partnership Knowledge Management and Dissemination, 2010). Educational research at the secondary level has shown that pedagogical content knowledge (PCK) for teaching modelling has several interrelated dimensions: knowledge of what makes good modelling tasks, ability to cognitively and conceptually analyse those tasks, recognizing common student difficulties, knowledge of appropriate interventions in students' work, assessing products of students' modelling work, and navigating institutional barriers related to transforming to a modelling-focused approach (Borromeo Ferri, 2018; Greefrath et al., 2021).

Using these principles, we created and hosted *A Master Class in Modeling the Life Sciences*, at Harvard University, in Cambridge, MA, which ran in 2023 and 2024. Each iteration of the workshop lasted one week and contained 35 contact hours. The workshops were facilitated by an interdisciplinary team: two mathematical biologists, a mathematics education researcher, and a university mathematics instructor specializing in active learning pedagogies. The workshops ran concurrently with Harvard's Summer School, a program for high school seniors to study a variety of advanced and interesting topics they may not get access to at their schools. The Summer School offered a class on the Mathematics of Biological Systems, using the LS30 materials, taught by two Harvard preceptors.

## REFINEMENT FROM YEAR 1 TO YEAR 2

In Year 1, we had assumed that faculty would benefit from the experience of teaching with a modelling-focused lesson. We had also assumed that, given their doctoral training in mathematics, that the mathematical content would be more straightforward for them than it turned out to be. The result was that the participants' PCK was low, and they did not learn as much from the teaching experience as we had hoped. Our takeaway lesson was that no matter how much math someone knows, they may not know dynamical systems in a way that bypasses algebraic manipulations and the more procedural aspects of calculus – that is, the way that students would understand it.

We responded to faculty's suggestions from Year 1 by changing the focus of the workshop. In Year 2, we built the workshop around engaging with the content of the LS30 materials through a combination of lectures and problem-solving sessions where participants could work through the pivotal learning objectives laid out through homework items and lab problems – the same ones that students at UCLA and in the Harvard Summer School would see. To focus on problem-solving, we removed the team-teaching experience. We also invited a GTA from UCLA and 3 faculty from universities that had already adopted the materials to share their perspectives. We distilled the mathematics and biology content to five key themes that run through LS30's approach: (i) The Art of Modelling (ii) Derivatives & Integrals from a Modelling Perspective (iii) Equilibrium Points & Stability (iv) Oscillations & Attractors (v) Chaos. Each day, for five days, participants read relevant chapters in the textbook, attended an interactive lesson on the mathematical and biological theme of the day, worked through the Python lab materials keyed to the lesson, and solved homework problems from the theme. The organizational team were available throughout to answer content or pedagogical questions. We also scheduled purposeful time for discussion, reflection, and building professional networks.

In a follow-up survey, faculty reported difficulties with concepts related to (Hopf) bifurcations, bistable systems, equilibrium points, and limit cycles & attractors as well as HPG systems, homeostasis, oscillation, and types of feedback in ecological systems. They reported that they were most familiar with "change equations" and ODEs, vector fields, the FToC, and classifying equilibrium points from both a mathematical and a modelling perspective. The most helpful sessions were the content lectures, observations of the live Harvard Summer School lessons, problem solving sessions, Python lab sessions, and the panel presentations from faculty teaching the course at other institutions. Participants reported they would have liked more time for sharing materials from versions of the course running at various institutions and more time to work through the student-facing activities with discussion about how instructors implement these. We infer that participants who have gone through the content portions of the workshop need additional practical guidance of "how this course looks" in the reality of the classroom. With respect to insights, two participants reported:

> "I didn't look at the book ahead of time, but I did work Lab 1. Based on the experience of doing that lab, I didn't get the sense that the course would really challenge my notions of what I thought a lower division math course could be. I was very mistaken. As a result, at every moment last week I was pondering the 'mathiness' of it all and letting my world expand."

> "my biggest insights on this came when Eric said 'this isn't just a better way to teach traditional calculus, this is a fundamentally different course based on what these students actually need.' As the week went on […] I came to better understand WHY these topics are actually more important (there are consequences for not understanding dynamical systems and modeling)."

We take these quotes as emblematic of the work that the LS30 materials and our workshop can do for changing faculty's outlook on the introductory mathematics experience for life science majors.

## CONCLUSIONS AND FUTURE WORK

Most workshops for professional development focus on innovating faculty pedagogy. While modelling pedagogies are important, we found that using LS30 materials poses special challenges to related to content. We were surprised at how difficult the concepts of limit cycles, attractors, bifurcations were. In fact, we encountered faculty reasoning hitting many of the same conceptual hurdles when translating between phase portraits and time series as we have observed with students. While faculty may be "quicker" at learning mathematics content than their students, learning new ideas can only be done through solving problems. Thus, we would caution others seeking widespread dissemination of their innovative calculus materials to build in time for faculty to "play" with the problems in the new curriculum. We found that the amount of contact hours was appropriate, that faculty benefit from working with simulations and the Python labs, which gave them confidence in using those non-traditional materials. In the future we will run a third version of the workshop with focus on curating a database of materials and pedagogical notes to share with interested faculty.

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# Role of some referents of Mathematics education in teachers' reflections on the causes of students' errors in university mathematics

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*Abstract. The author's doctoral research sought to characterize changes in a group of university mathematics teachers' conceptions of the causes of student errors. In the context of a mini-course on mathematics education, the participants reflected and discussed this topic, using theoretical references to analyze possible causes of errors. Some results are presented on the role of epistemological and didactic obstacles, semiotic transformations and misconceptions in the teachers' reflections.*

*Keywords: epistemological obstacle, didactic obstacle, semiotic transformations, misconception, change of conceptions, causes of errors.*

## INTRODUCTION

Research on teachers' conceptions and explanations of the causes of errors in mathematics reveals significant differences compared to the frameworks of Mathematics education that seek to explain them critically. While some teachers attribute errors to students' lack of interest and preparation (Gagatsis & Kyriakides, 2000), other factors pointed out include a lack of conceptual understanding, negative attitudes, deficiencies in prior education, traditional teaching, and reading problems (Ramírez, 2012). These explanations are based on teachers' beliefs about mathematics and its learning, which can negatively influence their understanding of the causes of errors.

According to Ball, Thames and Phelps (2008), a key component of expert teaching in mathematics is the ability to analyze the sources of errors and deepen the understanding of students' learning through errors. In this sense, it was fundamental for the study to examine how teachers can incorporate technical references from Mathematics education in their analysis of the causes of errors, which could help them to improve their understanding and approach and therefore generate changes in their conceptions.

## RESEARCH, THEORETICAL FRAMEWORK AND METHOD

In the doctoral research, the following question was answered: What changes in the conceptions of a group of teachers (in service) about the causes of errors in mathematics, of university students, occur from reflection, discussion and socialization, in specific focus group meetings, based on the analysis of theoretical referents of Mathematics education?

The research was conducted in two phases. In the first phase, a group of 25 in-service university mathematics teachers [1] answered the initial inquiry questionnaire. In this questionnaire, participants were asked to characterize, based on their own experience, some errors and their possible causes. Participants who expressed interest and commitment continued in the second phase of the research. In this stage, 4 focus groups were formed (3 members each); the teachers participated in a mini course of four sessions (1 session per theoretical referent). Each session included: an initial individual reflection on the possible causes of errors presented in student solutions to university mathematics activities [2]; an initial socialization of these reflections; the presentation of the theoretical referent by the researcher; and a final socialization to evaluate the relevance of the referent in the explanation of the errors analyzed.

The transcribed information was analyzed using an inductive-deductive categorization (Navarrete, 2011) for each focus group, for each session and for each participant. Taking into consideration the objectives of the research and the literature on error analysis (Gagatsis & Kyriakides, 2000; Peng & Luo, 2009), 5 a priori categories were defined for the study of the teachers' conceptions [3]. To establish changes in the participants' conceptions, the criterion suggested by D'Amore & Fandiño Pinilla (2004) and Alsina (2012) was followed, which consisted of comparing the teachers' conceptions before and after the training program. In our research, this comparison was carried out in detail before and after each of the focus group meetings.

The theoretical references studied were epistemological obstacles (Brousseau, 1976), didactic obstacles (Brousseau, 1976), semiotic transformations (Duval, 1993, 1999) and misconceptions (D'Amore, 1999). The term conception was adopted in the sense of Llinares (1996) as a range of teacher cognitions that includes beliefs and knowledge from experience. The change in conceptions was understood, according to Wilson & Cooney (2002), as their development or modification over time. It is assumed that changes in teachers' conceptions can help improve their teaching (Bobis, Way, Anderson, & Martin, 2016) and student performance (Wilkins & Brand, 2004).

## CONTEXT

The participants in the two phases of the study are in-service teachers who have taught different mathematics courses at Colombian universities in the city of Bogotá and in the municipalities of Chía and Soacha. In addition, the teachers have diverse training and experience [4]. The participants have most of their teaching experience in the following courses: Basic mathematics (also called Precalculus), Differential Calculus and Integral Calculus.

## RESULTS AND DISCUSSION

The teachers perceived as novel and significant the methodological potential of academically analyzing the causes of errors by means of referents from Mathematics education. Although they considered this framework pertinent (with some reservations) and showed partial attempts to incorporate it, a *natural* adoption in their analyses was not evident. This could be due to the fact that the participants are aware of the need to study these references in depth and be able to adopt them in their professional practice.

Some of the teachers recognize that their teaching decisions can hinder learning and even generate errors in mathematics for their students. This idea, conceptually aligned with the notion of didactic obstacle, emerges from the teachers' reflections based on their own experience. It is possible that this experiential familiarity facilitates the understanding of the didactic obstacle, in contrast to a potential lesser willingness to adopt the other theoretical referents addressed in the course. Epistemological obstacles, semiotic transformations and misconceptions are all new concepts for the participants.

Several of the participants in the different focus groups tend to interpret the semiotic transformations of conversion and treatment in terms of processes associated with problem solving and algorithmic processes respectively. In addition, some of the participants considered that problem solving is a critical aspect of mathematical learning that should be given special emphasis in mathematics class.

The misconception presented resistance from some participants because they considered that it did not properly explain the causes of the errors present in one of the tasks analyzed, or only partially explained them. Additionally, some participants recognize the existence of misconceptions in the students' conceptualizations (about mathematical concepts) that are also very difficult to remove.

Although some participants in their reflections refer to historical facts that highlight the complexity of the evolution of mathematical ideas, they do not fully recognize their potential influence on mathematical learning. This observation could explain the participants' difficulty in identifying the potential of the epistemological obstacle as a tool for understanding the causes of certain errors.

The study did not directly analyze the impact of changes in participants' conceptions on their classroom practices when addressing their students' errors. However, some professors drew attention to their difficulties in addressing student errors in depth in their classes due to time constraints imposed by the curriculum. This suggests a prioritization of adhering to the course schedule (or program) at the expense of a critical treatment of errors.

The socializations in the focus groups revealed several factors that shape the participants' conceptions about the causes of errors. These factors could explain some of the difficulties in incorporating the notions of Mathematics education presented in the meetings and the resistance to conceptual change. Some of these factors are not directly related to the theoretical referents studied but the teachers attribute them significant influence as potential causes of errors.

Participation in the study allowed teachers to become aware that a scientific approach to their students' errors in mathematics is possible and pertinent; it also enables some of these teachers to be willing to further their training in Mathematics education. In other words, it can be said that the participants have changed their conception of error and its causes from a vision that, despite taking into account various factors that complicate their analysis (such as context, comprehension problems, among others) and that did not contemplate an academic approach, has led to an understanding of error as a component of mathematical learning, which can be critically studied with tools provided by Mathematics education. Additionally, the study suggests that the changes in the participants' conceptions are located in an intermediate zone between those authors who consider that the change in conceptions is a gradual process (Guskey, 2003) and the position of authors such as Liljedahl (2010) who have found that, under certain conditions, rapid and profound changes are possible.

This study is aligned with international research on the specific knowledge of mathematics teachers about the causes of errors (Ball et al., 2008) and contributes to remedy the scarcity of studies in this area (Peng & Luo, 2009), which is especially critical in university mathematics. Finally, the relevance of deepening, in future research, the understanding of the impact of changes in the conceptions about the causes of errors on the practices of university mathematics teachers is emphasized.

## NOTES

1. These courses are part of university programs in which mathematics is a fundamental part of their training, such as engineering or administrative economics programs.

2. Student solutions to real activities (e.g., taken from assessments) were chosen. These solutions contained errors that can be understood and explained (at least partially) by means of the theoretical reference corresponding to the session.

3. A priori categories are: error identification, error explanation, willingness to incorporate the referents of Mathematics education in their analyses, conceptions about the causes of errors and other factors that influence the analysis of errors.

4. Among the participants, 10 have a master's degree (4 in Mathematics education, 4 in areas related to Education and 2 in other areas) and 2 have a PhD degree in Mathematics. In addition, at the undergraduate level there are 2 mathematicians, 5 bachelor's in Mathematics, 1 physicist, 1 bachelor's in Physics and 3 engineers. In Colombia, university bachelor's degrees focus primarily on training in pedagogy and didactics, enabling professionals to work as teachers.

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# Should the Cauchy-Riemann equations be named after d'Alembert?

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*The Cauchy-Riemann (CR) equations are a pair of partial differential equations (PDE) that occupy a prominent place in courses on complex analysis, since they form the necessary and sufficient condition for a complex function* $f(x+iy) = u(x,y) + iv(x,y)$ *to be complex differentiable. Interestingly, they are almost identical to the conditions for a 2D fluid to be incompressible and irrotational. This is not a coincidence, as there are common historical roots between hydrodynamics and complex analysis. In this contribution, aspects of this common origin are explored, and pedagogical opportunities are outlined. It is argued that such debates can enhance students' epistemological awareness of how mathematical knowledge is developed, especially concerning the fruitful interplay with physics.*

*Keywords: Cauchy-Riemmann equations, Complex Analysis, Fluid dynamics, D'Alembert.*

## IDEAL FLUID FLOW AND THE CAUCHY-RIEMMANN EQUATIONS

For the pure mathematician, it can be surprising to find out that the very first time a complex variable function appeared in history was in d'Alembert's groundbreaking work in fluid dynamics (D'Alembert, 1752). To understand this origin, we will first derive a pair of partial differential equations by imposing that a 2D fluid flow is both divergence- and curl-free. D'Alembert's original derivation of these equations are quite difficult to follow due to his unfamiliar notation and physical reasoning. The important thing to keep in mind is that it is based solely on physical assumptions. The interested reader can find details of d'Alembert's original derivation in Calero (2008, pp. 374-399).

Consider a planar fluid flow described by the velocity vector field $\mathbf{v} = \mathrm{u}(\mathrm{x},\mathrm{y})\hat{\boldsymbol{\imath}} + v(\mathrm{x},\mathrm{y})\hat{\boldsymbol{\jmath}}$, where $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are the usual unit vectors, not related to imaginary units (this "subtle" difference is crucial!). Suppose that the fluid is incompressible, meaning that the field has a vanishing divergence ($\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$) and irrotational, meaning that the field has a vanishing curl ($\boldsymbol{\nabla} \times \mathbf{v} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$), i.e., the field is conservative. Rewriting these conditions, we obtain a rather peculiar-looking pair of partial differential equations:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \tag{1.1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{1.2}$$

Aren't these the famous Cauchy-Riemann (CR) equations, which need to be fulfilled for the function $f(x+iy) = u(x,y) + iv(x,y)$ to be complex differentiable? Well, not quite, because of the minus sign. But it is still a remarkable "coincidence" that could be pointed out to invite students to reason about possible connections between complex analysis and fluid dynamics. Moreover, being aware of the first appearance of the CR equations reverses the order of the common narrative, where hydrodynamics appears as an application of complex analysis (e.g., complex potentials). But does that mean that the CR equations should be named after d'Alembert? The answer a bit nuanced, as discussed below.

**CLEVER TRICK TO SOLVE A PAIR OF PDES**

After expressing these physical conditions, the problem is purely *mathematical*: find the functions $u(x,y)$ and $v(x,y)$ that satisfy equations (1.1) and (1.2), given certain initial and/or boundary conditions. In essence, D'Alembert showed that satisfying these equations is equivalent to assuming that $vdx - udy$ and $udx + vdy$ are *exact differentials*, respectively, i.e.,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \leftrightarrow \quad vdx - udy \quad \text{is an exact differential} \quad (2.1)$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \leftrightarrow \quad udx + vdy \quad \text{is an exact differential} \quad (2.2)$$

Let us see why this is the case. By an exact differential we mean that there exists a (scalar) function of two variables $f(x,y)$, such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{3}$$

The other ingredient needed to understand this equivalence is the so-called equality of mixed partials, which asserts that interchanging the order of taking partial derivatives of a function does not affect the result. Mathematically, this is expressed by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{4}$$

Is the equivalence expressed in (2.1) and (2.2) clear now? Let us start with (2.1), i.e., assume that $vdx - udy$ is an exact differential. Following (3), $v = \frac{\partial f}{\partial x}$ and $u = -\frac{\partial f}{\partial y}$, which according to (4) yields Eq (2.1). This seems rather confusing but is actually simple! The reader is invited to check the equivalence (2.2).

Now we have a different, yet equivalent, way to formulate the same mathematical problem: find the functions $u(x,y)$ and $v(x,y)$, so that $vdx - udy$ and $udx + vdy$ are both *exact differentials*. D'Alembert's profound insight was to find a way to "gather" these two expressions in a "single" entity, that could be integrated. He did that by manipulating complex numbers.

## IMAGINARY PARTS DESTROY THEMSELVES

Here is his line of reasoning. If $udx + vdy$ is an exact differential, so is $udx + iv\frac{dy}{i}$, where $i$ is the imaginary unit.[1] Similarly, if $vdx - udy$ is an exact differential, so is (multiplying by $i$) $ivdx - iudy$, which is equal to $ivdx + u\frac{dy}{i}$. Adding $udx + iv\frac{dy}{i}$ and $ivdx + u\frac{dy}{i}$ results in $(u + iv)(dx + \frac{dy}{i})$. Now it is worth taking a moment to contemplate this last expression.

The manipulations of this last paragraph seemed arbitrary, pointless, perhaps even forbidden, but the end result is suggestive. D'Alembert managed to "gather" the two functions $u(x, y)$ and $v(x, y)$ in a single entity, the *complex* function $u + iv$. Thus, if $(u + iv)(dx + \frac{dy}{i})$ is an exact differential, then $u + iv$ is a function of $x + \frac{y}{i}$ (or of $x - iy$), which appears to be a complex variable function. Now, if instead of adding, we *subtract* $udx + iv\frac{dy}{i}$ and $ivdx + u\frac{dy}{i}$, we obtain $(u - iv)(dx - \frac{dy}{i})$, which leads to the assumption that $u - iv$ is a function of $x - \frac{y}{i}$.

For someone versed in modern complex analysis, it is hard to understand what d'Alembert is trying to achieve. Here is an opportunity to encourage students to discuss what d'Alembert is up to with these manipulations, and what would it mean at the time to treat $u + iv$ as a function of $x + \frac{y}{i}$ . The details in the original are a bit complicated, but, in essence, he is expressing the function in terms of its complex conjugate to get rid of the imaginary parts.

Notice that d'Alembert is assuming that $f(z) = \overline{f(\bar{z})}$. For Euler, this is the fundamental theorem of complex numbers (Klein, 1959, p. 627) and he used it to solve evaluate several (real) integrals. However, $f(z) = \overline{f(\bar{z})}$ is *not* generally true in modern complex analysis. This contradiction has great pedagogical potential. The crucial difference is that complex functions are not entities yet, and the whole point is to manipulate them so that the "imaginary parts destroy themselves". After all, physical quantities, such as velocity components of a vector field representing fluid flow, should always be real.

## COMPLEX NUMBERS HAD NO GEOMETRICAL MEANING

For real functions, we usually have a good geometrical understanding of $f'(3) = 2$, but what does, e.g., $f'(1 + i) = 3 + 2i$ mean? A powerful way to visualize the derivative of a complex function is the notion of *amplitwist* (Needham, 1997), which interprets the derivative as a local amplification and twist of *vectors* at a point in the complex plane.[2] If $f'(z)$ exists at point $z$, it means that every infinitesimal variation $dz$, from $z$,

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[1] D'Alembert wrote $\sqrt{-1}$ in the original. Of course, multiplying and dividing by the same number ($i$) does not change the expression, but at this moment it is perfectly fine to ask oneself what is the point of doing this.

[2] Wonderful animations of a complex derivative are found in https://www.youtube.com/watch?v=0CHZMY02Dhk and https://www.youtube.com/watch?app=desktop&v=b8_3PFjiJvY.

has the same corresponding scaling and rotating factor given by $df = f'(z)dz$. It can be shown that this only occurs when the CR equations are satisfied.

However, when we compare this with d'Alembert's (and Euler's) use of complex functions there is an important link missing, namely the geometrical interpretation of complex numbers as vectors in a plane, which came *only* in the early 1800's (Andersen, 1999). It is quite instructive to learn that complex functions appeared *before* the very geometrical interpretation of complex numbers!

Let us conclude by addressing the original question: should be CR equations be named after d'Alembert? Although it seems tempting to assert that studies on fluid dynamics were crucial for the development of complex analysis, they were probably not so influential. Indeed, it was only with Cauchy and Riemann that complex functions became entities, where complex differentiation and integration were formally defined (Bottazzini & Gray, 2013). This is in stark contrast with d'Alembert and Euler, for whom the complex derivative did not even make conceptual sense, mainly because complex numbers did not have a geometrical interpretation yet. Thus, the CR equations should *not* be named after d'Alembert.

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# STEM majors perceived value of an introductory calculus course-based research experience.

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*Almost all science, technology, engineering and mathematics undergraduates take undergraduate calculus, but the traditional applications of calculus in most textbooks are not equally relevant to all majors. We propose that incorporating a research experience where students must investigate and apply calculus in the context of their chosen field will make the content more meaningful. The purpose of this investigation was to determine to what extent such a project in a flipped introductory calculus class benefitted student participants. Overall, participants found value in the project, and felt that it and the discussion-based flipped classroom deepened their understanding of introductory calculus concepts.*

*Keywords: introductory calculus, perceived utility, course-based research experience.*

## INTRODUCTION

There has been a nationwide effort to update undergraduate STEM education, particularly calculus, to better align with how individuals learn (Ellis-Hagman, 2021; Viera, 2019). These efforts stress the importance of involving undergraduates in research (AAAS, 2011). Research shows that undergraduate research experiences significantly improve students' cognitive abilities, emotional well-being, and psychosocial development, leading to increased intention to pursue further education or careers in science (Lopatto & Tobias, 2010). To provide students with opportunities to get involved in doing research, approaches such as Course-based Undergraduate Research Experiences (CUREs) were developed (Wei & Woodin, 2011). The general characteristics of a CURE are a scientific approach to a novel problem, student-led discovery, iteration, collaboration, and a research contribution (Beck et al. 2023), though not all advocates of CUREs agree that the contribution needs to be in the form of a publication and argue that a discovery novel to the students is sufficient, particularly in CUREs in introductory classes (Dolan, 2016); we take the latter position for the purposes of introductory calculus. Dolan (2016) highlighted many benefits of CUREs, such as providing opportunities for students to make discoveries, improving persistence in STEM, and integrating teaching and research efforts. CUREs have been well developed in scientific fields, but there is a lack of Math CUREs in the literature.

The purpose of this study was to investigate the perceived benefits of a CURE in a modified introductory calculus class. Since almost all STEM majors take introductory calculus, we argue that it is paramount to make this course as relevant as possible for all majors, and research projects tailored to students' interests and majors may increase the relevance of calculus to STEM majors. The research question that guided our inquiry was: How do students perceive the value of a CURE project in comparison to other components of a student-centered calculus course?

## METHODS

In Fall 2023, there were two sections of Calculus I (16 and 28 students). In Spring 2024, there was one section with 13 students. Each section met five times a week for 50 minutes per class session. All three sections were flipped classrooms. Students were offered pre-recorded video lectures and materials by the professor before the meeting session. During the class meetings, students were split into groups of three to four to discuss problems prepared in advance with material covering the previous or current week(s). Then, students communicated their work on the problems with other groups or discussed them on the board with the rest of the class.

The semester-long research project aimed to involve students in working in groups to solve a problem from students' major using Calculus I concepts in at least one of the key steps in the solution. Students formed groups of three to four individuals from the same major to work on a project problem. Each group chose or created their own project problem. For example, the project 'Water Level Management System for H2O Mermaids Co.' aimed to design an automatic system for monitoring and controlling the water levels in a spherical reservoir to provide a reliable supply during emergencies. Students calculated the volume of a spherical reservoir by using the integration concept to determine key water levels. With the Pythagorean Theorem, applying the Method of Disks, a Calculus II concept (easily understood by Calculus I students), students obtained the area of vertical slices of the tank to calculate the volumes required for controlling water collection. In another project, 'Analysing RL Circuit Dynamics,' students showed that by applying Calculus concepts, one can better understand electrical systems. They explored how 'current' changes over time in RL circuits by studying an elementary differential equation derived from Kirchhoff's voltage law and then solving it using 'separation of variables' and integration. Students also calculated how the circuit responds at specific intervals.

The theoretical framework for this study was situated expectancy-value theory (SEVT), which posits that student motivation is a combination of a learner’s perceptions about expectations for success, their ability, the value of the task, and any negative evaluations (costs) of the task (Eccles & Wigfield, 2020). This inquiry primarily focused on the task value and cost components of the framework. Based on our prior research with students’ perceived value of course components (Celik & Dibbs, 2024), we anticipated that most of the perceived value would be on lectures and discussion, with the project receiving less perceived value overall because it extended rather than supported the learning of students.

At the end of each semester, participants were invited to participate in a semi-structured interview about their perceptions of the course components based on the components of SEVT. Seven participants, five male and two female, chose to participate in the interviews across the two semesters. Their majors were electrical engineering (3), mathematics (2), chemistry (1), and computer science (1). After transcribing the interviews, we parsed the participant responses into our unit of analysis, a statement, so that each statement represented a single idea. These statements were then assigned three levels of value codes. The first code assigned to each statement was a domain code, which identified which course component a participant was discussing. The second code assigned to all statements was a value code; the statement was characterized as positive, neutral, or negative regarding the course component. For example, the statement "My project was about controlling the water level in a water tower," would have been coded as (project, neutral). For the purposes of this paper, we did not report neutral statements due to length limitations, but all of the neutral statements consisted of factual descriptions of course components without value judgments attached. If a student made an explicit statement that a course component supported their learning, a third emphasis code called support was added to the code of the statement. The two authors initially coded all statements independently and then reconciled the codes to 100% agreement through discussion.

## FINDINGS

The project was the most discussed course component in the interviews (Table 1), and all seven participants spent significant interview time discussing the project. The neutral comments in the analysis were elevated because all participants gave a description of their project topic without a value judgment; without those statements the positive and neutral comments about the project are approximately equal. Interestingly, the project was tied for the second most support codes among the course components with instructor actions (things the instructor did, like asking questions during class to facilitate learning) and behind only watching the instructor's lectures.

| **Domain Code** | **Positive** | **Neutral** | **Negative** | **Total** | **Support Total** |
|---|---|---|---|---|---|
| Project | 9 | 18 | 5 | 32 | 4 |
| Other Assignments/Recitation | 7 | 12 | 2 | 21 | 0 |
| Discussion | 8 | 8 | 3 | 19 | 1 |
| General | 9 | 9 | 0 | 18 | 2 |
| Lectures | 6 | 11 | 0 | 17 | 4 |
| Instructor Actions | 8 | 0 | 1 | 9 | 5 |
| Office Hours | 0 | 0 | 1 | 1 | 0 |
| Total | 47 | 58 | 12 | 117 | |

**Table 1: Summary of participant codes**

All participants made at least one positive statement about the project; with two major themes emerging. Other than noting the project helped participants learn current material, positive comments also indicated that the project had two main benefits. Three participants felt that their project helped them relate calculus to the real world, which in turn helped them find the material more relatable, such as Participant 4 in her interview:

> It put it in a real-world scenario, which is a very different perspective, which helped a lot. And I've noticed that if a teacher will explain how a calculus problem or a math problem in general could be used in real life, it gives a different perspective on how, basically, why we're learning this. And that helped a lot, understanding how there is connection, even though it doesn't seem like there's any connection at the moment.

The remaining participants found more immediate benefits; their project helped them see the relevance of calculus to their major, such as Participant 7, a computer science major, in his interview:

> Yeah, I guess it gave me, I already understood how math plays a part into things. I know how it's included in the sciences. It's contributed to all this stuff, but it also, for us, we focused on calculus and computer graphics. I guess it taught me specifically that it's just that we take things for granted. If it wasn't for calculus, they wouldn't have been able to develop software that immediately calculates interactions between objects or graphics, especially.

Only three participants made a negative statement about the project, even with probing. Four of the five negative statements were about having to limit the scope of the project because of limited knowledge in the major, as Participant 3 explained in his interview:

> We really wanted to do an actual electrical engineering problem instead of more of a finance problem. But we, you know, we're kind of, kind of right at the beginning. So we're not super familiar with this, with like electrical engineering stuff like, like how derivatives are involved in it and all this other stuff. So we kind of had to come to the compromise of a finance problem.

## DISCUSSION

Overall, participants saw a surprising amount of value and limited costs in the research project embedded into introductory calculus. Participants felt that the project supported their learning as much as the lectures, and only explicit teaching by the instructor was seen as making a larger contribution to their learning throughout the semester. Participants found that the project helped them relate to later material in the course, relate calculus to their major, and relate calculus to situations they would likely encounter in their future careers. Even most of the negative comments indicated some support for the project, since participants would have preferred to do a project larger in scope that related even more closely to their chosen field, but were limited by their current knowledge.

Although our research project does not strictly fit the definition of a CURE, we argue like Dolan (2016) does; in introductory calculus it is sufficient that students make connections about mathematics beyond that of the standard curriculum to introductory calculus topics. All participants found at least some positive benefit from their research project, regardless of achievement level. Henderson and Kose (2018) suggested that participation in CUREs benefits average to lower achieving students the most, and our findings tend to agree.

There are few limitations to our inquiry: the number of participants is fairly small, and not as diverse as we would prefer. Our findings are significant because there have been few CUREs in introductory mathematics courses, and participants in previous CUREs reported their satisfaction with the CURE experience in surveys rather than in interviews. Future work on mathematics CUREs should focus more on refining a model for CUREs in the calculus sequence, possibly in multivariate calculus.

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# Students Concept Mapping the Derivative & Novel Tools for Analysis

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*Derivatives are a key concept in both single variable and multivariable calculus that are vital for students across disciplines to understand. Rates of change arise in many fields and everyday life. Researchers have been studying how students think about the components of the derivative for decades, but there is less known about the connections students make between these components. In this study, university calculus students created concept maps–visual representations of connections–which we analyzed using homology groups. We argue that homology is an innovative and useful tool for analyzing concept maps, complementing previous analyses conducted via qualitative techniques or scoring systems.*

*Keywords: Concept maps, derivatives, students' practices, topological data analysis.*

Derivatives are a key concept in calculus 1 (single variable calculus) and calculus 3 (multivariable calculus). Sometimes calculus 3 students say they remember derivatives but that they "never understood it and don't think it's important." This statement is from a calculus 3 students' reflection survey after a derivative activity. Since rates of change appear in all parts of life, such as the velocity at which cars travel or how quickly a medication breaks down in the body, they are essential for calculus students across disciplines to understand.

To examine the different ways in which derivatives can be thought about and how they are connected, we turn to Zandieh's (2000) and Wagner et al.'s (2015) frameworks. Zandieh's (2000) framework can be used to analyze the conceptual understanding of derivatives, and Wagner et al.'s (2015) extend the original framework to aid in analyzing the computational understanding of derivatives—two types of understanding necessary for students to succeed in mathematics courses (Rittle-Johnson, 2017). Successful students see mathematics as connected ideas (OECD, 2013). Concept maps, "diagrammatic representations of 'meaningful' relationship between concepts" (Watson, 1989), can be used to visualize these connections students generate at one point in time. Concept maps have been used since the 1970s (Novak & Cañas, 2006), in early childhood education (Birbili, 2006) through higher education (Mitra et al., 2023). Despite this, there is little recorded usage of concept maps in mathematics education (Evans & Jeong, 2023). Concept maps are often analyzed both qualitatively and quantitatively or strictly qualitatively (Asl & Koca, 2004). The quantitative analysis method that is often used is some kind of scoring system, of which there are multiple (Ekinci & Şen, 2020). This study aims to expand the use of concept maps in mathematics education and investigate new ways that mathematics education data, specifically concept maps, can be analyzed. The new method offered here, analyzing via homology groups, is a method that is part of the broader idea of topological data analysis (TDA). Homology groups, described below, and TDA process high-dimensional data and give insights into the shape of data (Taiwo et al., 2024).

## RESEARCH QUESTION

What insightful information about students' connection-making do homology groups give when used to analyze student-generated concept maps?

## METHODOLOGY

This study was conducted at a large R1 public university in the Southeastern United States. Data was collected from students who were enrolled in calculus 3 in 2024. These classes are typically taught in a 2-2 lecture recitation format. That is, twice a week ~250 students attend a lecture with the instructor of record and the other two days, students meet with a graduate teaching assistant in classes that have ~32 people. Some of these classes are also taught in a strict "lecture" format, meaning that four days a week, the entire class meets with the instructor of record. These classes usually have 90-120 students and are taught by faculty.

The data was collected at a supplemental instruction session that students attended voluntarily. At this supplemental instruction session, students created a concept map using up to 23 terms from a word bank. These 23 terms can be seen in Figure 4, and are derived from Zandieh's (2000) and Wagner et al.'s (2015) theoretical frameworks. In this report, three examples of student-generated concept maps are analyzed, illustrated in Figures 2, 3, and 4. To reference the creators of these concept maps, we will use the aliases Anna (Figure 2), Emma (Figure 3), and Lisa (Figure 4). Note that these aliases do not necessarily correspond to the gender of the creator.

The concept maps were analyzed using homology groups, a mathematical tool to detect "holes." *Homology* is an idea from algebraic topology; using abstract algebra to study topological spaces (Hatcher, 2001). These groups measure the dimensionality of the connections displayed in the concept maps and give information about the disconnectedness. Homology is denoted as $H_i$ where $i$ is the dimension. It is computed over the set of all integers, $\mathbb{Z}$. The vertices/terms are the objects that make the zeroth dimension. The edges/connections are the objects that make up the first dimension. The triangles formed by three adjacent vertices and the three edges that connect them make up the second dimension. The tetrahedrons formed by four triangles make up the third dimension. Examples of these can be seen in Figure 1. This continues as higher dimensions are considered.

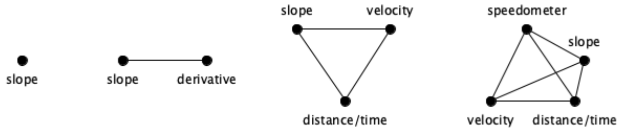


**Figure 1: Objects in zeroth, first, second, and third dimension (read left to right).**

## RESULTS

While we computed the homology groups algebraically (Hatcher, 2001, p. 105) of each concept map, here the homology of the concept maps is discussed geometrically. In the zeroth dimension, $H_0 = \mathbb{Z}^n$, $n$ is the number of connected components in a concept map. In the first dimension, $H_1 = \mathbb{Z}^m$, $m$ is the number of cycles present that have more than 3 vertices, i.e. not triangles. We call these “holes.” These are the only homology groups present in the three concept maps considered.

Figure 2 shows Anna’s concept map that is one component; that is, all vertices are connected, you can reach one vertex from any other by following along the edges. Figure 2 is made up of 19 vertices, 24 edges, and has no holes. It has $\boldsymbol{H_0} = \mathbb{Z}$, $\boldsymbol{H_1} = 0$, $\boldsymbol{H_2} = 0, \boldsymbol{H_3} = 0$, and $\boldsymbol{H_4} = 0$ .

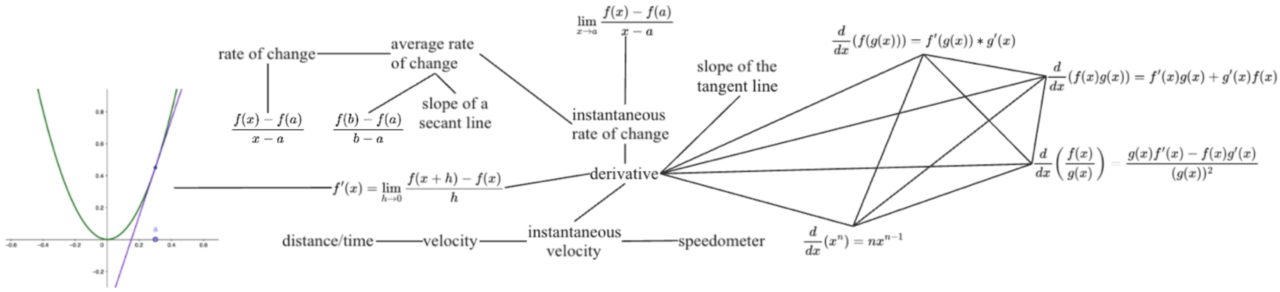


**Figure 2: Anna’s single component example.**

Figure 3 shows Emma’s concept map that is two components and has no holes. It is comprised of 14 vertices and 12 edges. It has $\boldsymbol{H_0} = \mathbb{Z}^2$ and $\boldsymbol{H_1} = \mathbf{0}$.

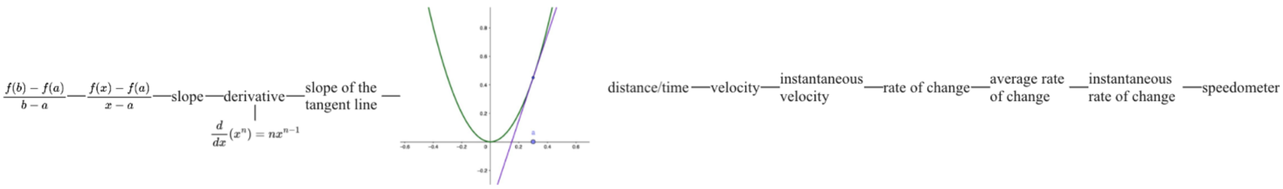


**Figure 3: Emma’s two-component example.**

Figure 4 show’s Lisa’s concept map that is four components and has one hole. It is comprised of 23 vertices, 26 edges, and six triangles. It has $\boldsymbol{H_0} = \mathbb{Z}^4$, $\boldsymbol{H_1} = \mathbb{Z}$, and $\boldsymbol{H_2} = \mathbf{0}$.

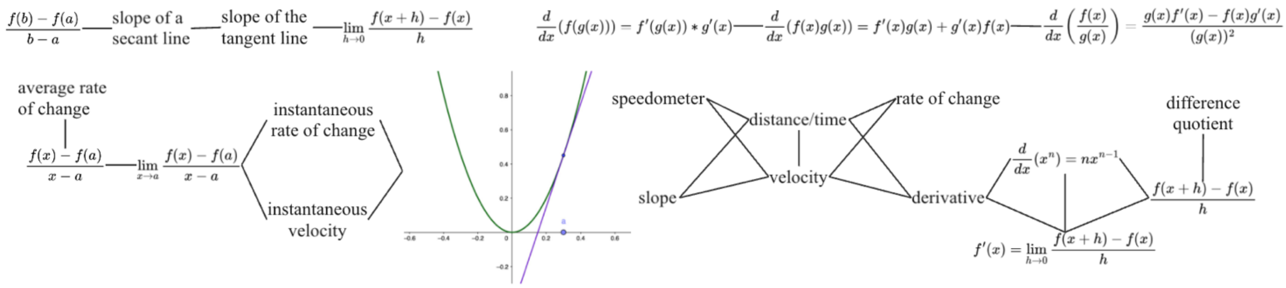


**Figure 4: Lisa’s four-component example.**

## DISCUSSION AND CONCLUSION

Each concept map has different homology groups in the zeroth dimension. This demonstrates how varied the complexity of connections can be, even when examining only three students’ concept maps. At present, we have no data to suggest that certain homology groups arising at varying stages of the learning process are better for learning.

Anna having $\boldsymbol{H_0} = \mathbb{Z}$ means that all the terms she considered are connected in some way. Emma has double the number of components ($\boldsymbol{H_0} = \mathbb{Z}^{\mathbf{2}}$), and even still, Lisa has quadruple the number of components ($\boldsymbol{H_0} = \mathbb{Z}^{\mathbf{4}}$). Emma and Lisa having more distinct components indicates more disconnectedness across all terms, but it does not give information on the connectedness within each individual component. To learn about the connectedness within each component, we can turn to other homology groups, including $\boldsymbol{H_1}$. $\boldsymbol{H_1}$ gives the number of holes. Holes represent where connections could have been made (to form triangles) but weren't. Anna and Emma have $\boldsymbol{H_1} = \mathbf{0}$ (no holes) while Lisa has $\boldsymbol{H_1} = \mathbb{Z}$ (one hole). This gives us much more insight into the connections students identify. We can see from this calculation that Lisa has instances where more connections could have been made. To learn even more about the connectedness within each component, we can turn to even higher dimension homology groups, $\boldsymbol{H_n}$ with $n > 1$.

Anna was the only student who made connections in any degree larger than two. She has $\boldsymbol{H_4} = \mathbf{0}$, which shows she made connections in the fifth dimension but had no "holes" in that dimension. Emma and Lisa both lack higher dimensional connections. It is expected that an expert's concept map would be more like Anna's in terms of the dimensions of connections that are made. All three concept maps examined here also used a different number of terms. This gives even more insight into what students deem to be valuable information about the derivative. It's possible that Emma only used 14 different terms because these were the most important to her at the time. Furthermore, Lisa used all the available 23 terms because she thought all of them gave insightful information.

This demonstrates that one can take many different perspectives when examining the information arising from a homology group analysis. This initial analysis using homology opens many doors for future exploration and shows that homology is an innovative and useful tool for the analysis of concept maps. It begs the question, "is there an ideal $\boldsymbol{H}_n$ for learning?" In the future, more information about the homology groups of concept maps could help establish if there is one type of connection that should be prioritized to optimize university calculus students' learning. It could also give insights into whether students in different disciplines could benefit from having different homology groups in their concept maps based on what is important in their chosen fields. For example, mathematics majors may benefit from connecting the symbolic formalism while engineering majors may have more use for the connections between the applications of the derivative like velocity and acceleration. When the homology groups and objects (vertices/terms, connections, etc.) in each dimension are considered in their totality, information about what types of connections are present in students' minds may be gleaned, allowing for trajectories of growth to be identified. This leads us to believe that other TDA methods, like persistent homology, may be useful in future analyses of student-generated concept maps.

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# Students' reasoning about dynamic rate of change visualizations

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*In this paper, we explore undergraduate engineering students' covariational reasoning in dynamic rate of change tasks that include linear dynamic animations and focus on how students elicit information from the animations. Eleven undergraduate engineering students' responses on two tasks about rate of change along with think-aloud and stimulated recall interviews are analyzed.*

*Keywords: rate of change, covariational reasoning, dynamic visualizations.*

## INTRODUCTION

Rate of change is a foundational concept in calculus (Thompson, 1994). An understanding of the concept of rate involves creating an image of change in relevant quantities, coordinating changes in both quantities, and creating an image of the simultaneous covariation of the quantities (Carlson et al., 2002; Thompson, 1994). To capture students' understanding of rate of change (and many other calculus concepts), covariational reasoning is highly relevant. Covariational reasoning is defined to be the cognitive activities involved in coordinating two varying quantities while attending to ways in which they change in relation to each other (Carlson et al., 2002, p. 354).

Previous research reports show students' difficulties in reasoning about rate of change in different settings such as dynamic situations, graphical contexts, and non-kinematic situations. In graphical and non-kinematic situations respectively, Jones (2017) and Rodriguez and McAfee (2023) report that students invoke time as a quantity in rate of change problems even when time is not involved. Jones (2017) also noted that some students experience difficulties in interpreting the derivative values and interpret them as values of the quantities rather than as rates of change. In graphical contexts, students pay attention to patterns of the graphs, interpreting rate of change as graphs' steepness while ignoring the axes (Rodriguez & McAfee, 2023). The researchers point out the need to use diverse contexts in rate of change tasks to improve students' comprehension. In dynamic situations, Carlson et al. (2002) found that even high-achieving students struggle with reasoning about rate of change. Carlson and colleagues suggested that physical enactment of dynamical phenomena could present opportunities for students' reasoning and understanding of rate of change and other concepts in calculus. Generally, many researchers have pointed out that over-reliance on static images and algebraic approaches for presenting dynamic relationships in calculus limits students' comprehension (Hong & Lee, 2022).

## THEORETICAL BACKGROUND

The concept of covariational reasoning (cf. Carlson et al., 2002) is central in this paper. In addition, we draw upon Elby's work (2000) who argues that the context may cue activation of some intuitive knowledge elements in learners, which Elby refers to as what-you-see-is-what-you-get (WYSIWYG), e.g. a horizontal line means stillness which may be correct in some situations but can lead to misconceptions in other contexts (such as in velocity-time graphs). The intuitive knowledge elements such as WYSIWYG may be activated due to compelling visual attributes in a context. Ainsworth (2006) notes that the learners need to learn to ignore the intuitive knowledge to interpret representations.

## METHODS

This paper draws upon a research project that employs multiple methods including think-aloud protocol, stimulated recall interviews, and eye-tracking[1] for exploring students' covariational reasoning in visualization-based tasks. Eleven undergraduate engineering students solved 12 multiple-choice tasks each on different aspects of rate of change. In this paper, we report on students' work on two tasks that included linear dynamic visualizations, which could be paused and replayed but did not allow for the manipulation of variables (see the tasks below). The students solved each task in silence on a laptop while their eye movements were tracked. They explained their reasoning aloud after clicking on one response option of their choice. The stimulated recall interviews were conducted afterward. Each student is assigned a unique number (P1–P11) and we refer to their respective numbers in the text.

The two tasks, cannon cow and frozen yoghurt, are adapted from Calculus Videos Project (https://calcvids.org/videos/). The cannon cow task (Figure 1, left) depicts a cow being shot from a cannon. The cow is shown to move upwards after

[1] Note that eye-tracking data is not included in this version as we are in initial stage of analysis.

being shot, reaching a maximum height before descending. On its journey downwards, a parachute opens, and the cow's speed is reduced from this point to the ground. The task asks about the rate of change of height with respect to the distance covered (option *a* correct response).

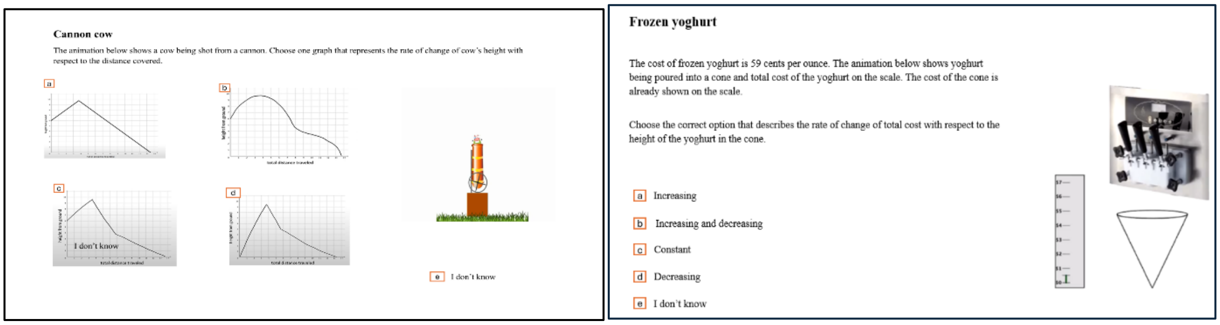


**Figure 1: The two tasks, © 2018 Copyright: Calculus Video Project.**

Frozen yoghurt (Figure 1, right) task depicts an animation of yogurt being poured while displaying its cost on a scale. The task asked about the rate of change of the cost of yoghurt with respect to its height that is increasing.

## RESULTS

**The cannon cow task:** All students incorrectly assumed that the task asked about the speed of the cow where they invoked time as an independent variable. P2 and P6 had selected the correct graph (option a) first, but later changed to other options. When they were prompted in the stimulated recall, P2 demonstrated an understanding that time was not involved in this situation:

> P2: When I look back at it. I think it should have been [option] a … this is just distance traveled. There is no speed involved. So even though the parachute slows it down. It just goes up and down …

P6, however, remained unsure. P6 had difficulty in engaging with the graph with distance on horizontal axis and was also confused between total distance covered and displacement.

> P6: because in my mind, if you have distance travelled, he [cow] only moves in one direction … but not that you go forward and backwards, then it becomes zero.

P1 and P9 considered apparent and intuitive features of the graphs and they discarded the graphs that were "pointy".

P1: Well, it was really that I imagine it like, for example in Physics 1, where we have diagrams of throwing balls and such things… so it was really just that I had never seen such a sharp point.

P9: this curve [option b] represents that better than steep lines … Because it doesn't just go straight up and down like that but moves more in a kind of harmonic motion.

Other students considered the moment when the parachute opened during the cow's downward motion and examined the position of the humps in the graphs for options b, c, and d intuitively while comparing these three choices. When prompted, none of them were able to conceptualize how time could be excluded from this situation.

**The frozen yoghurt task:** Six students answered correctly that the rate of change of the cost is increasing. Among these students, some (e.g., P7) reasoned based on the animation while others (e.g., P8) did not.

P7: I did not actually look at the animation so much, I just know that like as the height increases the amount of yogurt needed to add height also increases.

P8: ... as you can see at the start of the animation the scale goes very slowly and then it sort of takes off.

Three students (P1, P3, and P4) incorrectly chose the option b *increasing and decreasing*. Their reasoning showed that they only looked at the cone being filled up in the animation. P1's reasoning can be seen below:

P1: I thought that in the video, they specifically show that at the end they pull the handle up, and then I think back to when I use the ice cream machine when I pull the handle up, it comes out smoothly

P1: I didn't pay much attention to it…since they followed each other … Yes, it could perhaps have been increasing …

P1, P3, and P4, were cued by the visual of the cone being filled up speedily at start than the end and started thinking of the rate of the change of height of yoghurt with respect to time. From the rest, P6 chose *decreasing* but was uncertain if it could be increasing. P11 answered *decreasing* and had difficulty in grasping the quantities in question.

## DISCUSSION AND CONCLUSION

The cannon cow problem that was set up in a kinematics context, but did not ask for a rate of change involving time, was difficult to grasp for the students. They invoked time and thought of the rate of change of height with respect to time, a persistent issue reported previously (cf. Jones, 2017; Rodriguez & McAfee, 2023). Half of the students solved the frozen yoghurt problem correct, some leveraging the animation for their reasoning. Those who had it incorrect misidentified key quantities, influenced by the animation of yogurt pouring, which misactivated intuitive knowledge of ice cream machines (and how a cone fills up) and shifted focus to the wrong quantity (height and its rate of change). It remains unclear whether the cannon cow animation (parachute and speed reduction) prompted speed-related reasoning (Elby, 2000) or whether the issue stemmed from limited engagement with non-time-related rate of change problems. Eye-tracking data provide further insights into students' attention in these animations.

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# Teaching Multivariable Calculus to Biology Students

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*How can we teach multivariable calculus concepts to biology and life sciences students in a contextualized way? And what concepts are actually used in modern biology research? In this paper I present the development of a new Calculus for Life Sciences course and focus specifically on the part that introduces multivariable calculus concepts. The course focuses on modeling and studying the long-term behavior of dynamical systems arising in the life sciences. I show how the concept of linearization is central in biology as it is necessary to determine the long-term behavior of non-linear models that are so prevalent in the life sciences.*

*Keywords: calculus for life sciences, novel approach to teaching calculus, modeling, dynamical systems, multivariable calculus.*

For more than two decades there have been high profile calls from professional associations to change the teaching of calculus for biology and life science students (e.g., National Research Council (NRC), 2003; Association of American Medical Colleges and Howard Hughes Medical Institute (AAMC), 2009). These reports recommend the development of mathematics courses that focus on modeling with an "emphasis […] not on the [mathematical] methods per se, but rather on how the methods elucidate the biology" (NRC 2003, p. 170). The goal of such courses "should be to see biology in a whole new light as a result of the mathematical/computational approach to the subject" (NRC 2003, p. 170). The AAMC moreover explains that it expects medical school applicants to be able to "quantify and interpret changes in dynamical systems" (AAMC, 2009, p. 24). Over the last three decades courses and course activities have been developed to bridge the gap between mathematics and biology (e.g., Ledder, 2008; Robeva et al., 2022). While a variety of approaches have been described in the literature, I have found no article precisely describing how a multivariable calculus course could be completely transformed to truly interweave mathematics and the life sciences.

In 2013 the University of California Los Angeles (UCLA) started developing a new two-quarter "Calculus for Life Sciences" course (Garfinkel et al., 2017). The goal of this new course is to bridge the gap between the mathematics and life sciences and to introduce students to mathematical concepts actually used in modern biology research. By focusing on modeling and the dynamical systems arising in biology and the life sciences, the course follows the recommendations advocated in the aforementioned reports. In the first quarter of the course, which is described in detail in Bennoun and colleagues (2023), students start by learning how to write differential equation models that describe the evolution of biological systems. They then learn the concepts of derivative and integral. Finally, using graphical methods such as simulating time series, state space trajectories, and vector fields, students learn how to study the long-term behavior of (mainly) one-dimensional models. They are also introduced to the biologically important concepts of qualitative change, which mathematically corresponds to the notion of bifurcation, as well as oscillatory behavior in biological systems. In the second part of the course, the focus of the present paper, students acquire more advanced tools from calculus to determine analytically, and not only graphically, the long-term behavior of models. Since the second half of the course focuses on two-dimensional models, students learn concepts from multivariable calculus. The present article consitute an answer the question of how one can transform a multivariable calculus course in order close the gap between calculus and the life sciences. In the following sections I start by describing why students first learn about discrete-time models. I then explain why the study of the long-term behavior of two-dimensional models requires the notion of linear approximation in higher dimensions. Next, I show how studying long-term behavior naturally leads to the notion of qualitative change, or mathematically, of bifurcation. Finally, I briefly describe the impact of the course on student motivation and subsequent performance.

## COURSE CONTENT

As mentioned above a fundamental question for biologists is to determine the long-term behavior of biological systems. In the course we investigate questions such as: will two species competing for resources in a given area coexist or not? And if so, how many individuals of each species will survive in the long run? In the context of the biochemical process called glycolysis, which is one of the important ways cells have to obtain energy by breaking down sugar, how will the concentrations of chemicals involved in this process evolve over time? Will they eventually tend to specific levels or will they continue to oscillate indefinitely? In order to answer these questions, one needs to find the equilibrium points of the model and determine their stability. In a two-variable differential equation model, the equilibrium points are the points for which both equations are equal to zero. The stability of an equilibrium point determines whether the system is (locally) attracted to the point (a "stable" equilibrium point), repelled by it (an "unstable" equilibrium point), or not attracted nor repelled by it (a "neutral" equilibrium point). As such, it is the equilibrium points and their stability that "control" the long-term behavior of a model. It is thus crucial for students to develop the ability to find these points and determine their stability.

### Setting Up the Stage with Discrete-Time Models

In the second quarter of the course, students first learn about discrete-time models. These models are well-suited to study animal populations that have well-defined breeding seasons as well as other phenomena for which data comes in discrete times such as heartbeats. In particular, we study models defined by Leslie matrices. In such models, the new populations $\mathbf{X_{N+1}}$ are found by applying the matrix A defining the model to the vector of current populations $\mathbf{X_N}$, in other words we have $\mathbf{X_{N+1}} = A\mathbf{X_N}$. In order to study these models we introduce the different concepts necessary to define the notions of eigenvector and eigenvalue. Specifically, students learn about vector spaces, bases, linear transformations and matrices. While these notions are obviously not calculus concepts, the reason I mention them here is that eigenvalues are crucial to study continuous-time models in two or higher dimensions. In other words, when analyzing continuous-time models calculus and linear algebra notions become intimately linked and one cannot use purely analytic concepts to study such models. This part on discrete-time models can therefore be thought of as a mean to introduce the concepts of eigenvalue and eigenvector in a contextualized way.

**Stability of Equilibrium Points and The Importance of Linearization**

After the section on discrete-time models, we turn to continuous-time models, which is really the central part of the course. How one can determine the stability of an equilibrium point depends on whether the model is linear or not. If the model is linear, which means that the equations can be written as linear combinations of the variables, then it can be written in matrix form. In this case the stability of the equilibrium points is determined by computing the eigenvalues of the matrix that defines the model. However, the vast majority of the models arising in the life sciences are *non-linear* and therefore this "simple" method cannot be used. The question is thus how to analyze the non-linear models that are so prevalent in biology.

At that point in the course we return to the idea of using the linear approximation of a function to determine the stability of the equilibrium points of a model. This idea was first introduced for the one-variable case in the first quarter of the course and we now extend it to two-variable vector fields. To do so, students first learn about functions in two variables and how their graphs are surfaces in the three-dimensional space. Then, we introduce partial derivatives and study how they correspond to the slopes of specific lines tangent to the surface defined by the function. We can next move to finding the linear approximation of a vector field. The two equations of a model define a vector field $V: R^2 \to R^2$ that we write as $V(x, y) = (f(x,y), g(x, y))$, where $f$ and $g$ are two-variable functions. The linear approximations of $f$ and $g$ are thus the tangent planes in the three-dimensional space. We can combine the two linear approximations to find the equations of the linearized vector field, in other words, the first component of the linearized vector field is the linear approximation of $f$ while the second component is the linear approximation of $g$. Importantly, students learn that the linear approximation of a vector field can be written in matrix form and that this matrix is called the Jacobian matrix. The derivative of a two-dimensional vector field is thus conceived as the linear approximation of the vector field. Finally, students learn about the Hartman-Grobman theorem which states that in almost all cases linearizing a vector field preserves the stability of the equilibrium points. This result is really crucial as the vast majority of models arising in biology and the life sciences are non-linear. We then study and determine the long-term behavior of two-variable models such as competition models, predator-prey models, and the glycolysis model presented above. Importantly, students are always asked to explain what their results mean in biological terms.

### Qualitative Change and Oscillations

The study of the long-term behavior of models naturally leads to the important notion of qualitative change. One easily notices that the long-term behavior of a model can be qualitatively different depending on the value of one or several parameters. For example, in a competition model a change in the value of a parameter can determine whether the two species will coexist in the long run or whether one of the two species will go extinct. In the case of the Holling-Tanner model, which describes the evolution of predator and prey populations, the change in one parameter will determine whether the two species will tend to specific numbers or will keep oscillating with time. Mathematically, qualitative changes are bifurcations and they correspond to a change in the number or stability of the equilibrium points. Since the stability of the equilibrium points is "controlled" by the eigenvalues of the system or of its linearization, studying bifurcations essentially comes down to determining how the eigenvalues evolve as a given parameter changes. A particularly important type of bifurcation we study are called Hopf bifurcations. These bifurcations correspond to the appearance or disappearance of stable oscillations. Using this concept, one can explain why the concentrations of fructose-6-phosphate and adenosine diphosphate that are involved in glycolysis either tend to specific concentrations or tend to stable oscillations depending on the value of a parameter. Similarly, we can explain the appearance or disappearance of oscillations in the Holling-Tanner model.

### Course Impact on Student Motivation and Performance in Subsequent Courses

One natural question is whether by this new contextualized version of the course increases students' interest in the life sciences. In a survey conducted with students who had taken the old version of the course, 83.7% of the 289 respondents either disagreed or strongly disagreed that they were more interested in studying biology after having taken the course. By contrast, 78.3% of 332 students who had taken the new version of the course either strongly agreed or somewhat agreed that they had become more interested in science as a result of the course. As the vast majority of these students want to major in biology or the life sciences, these results suggest that the new Calculus for Life Sciences helps increase students' interest in their major whereas the old version was decreasing it.

Another question is whether the new curriculum adequately prepares students for subsequent science courses. Sanders O'Leary et al. (2021) studied this question by comparing the performance of students who had either taken the old or new version of the course. Using multilinear models, they compared grades in courses in chemistry, life sciences, and physics that are typically taken after the Calculus for Life Sciences series. For all three courses they found that having taken the new version of the course positively impacted students' performance in subsequent science courses.

## CONCLUDING REMARKS

The course developed at UCLA provides an example of how multivariable calculus concepts can be taught to biology students using authentic models and problems from biology. The crucial Hartman-Grobman theorem shows the importance of the concepts of linear approximation and eigenvalue for biologists. This suggests that when teaching the notion of derivative to biology students, it is really the idea of linear approximation that needs be underlined. It also shows how notions from calculus and linear algebra are intimately linked when working with two-dimensional models.

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# The evolution of the concept of infinitesimal and differential in physics through the analysis of university textbooks

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*This study examines 17 university physics textbooks with the aim of analysing how infinitesimals and differentials are treated and investigating if and how the approach to these concepts has changed over the years. What seems to emerge is that, as physics textbooks embrace the change introduced by Cauchy and Weierstrass in mathematical analysis by pursuing rigor, they increasingly marginalize mathematical definitions of calculus concepts and attribute to mathematics a technical rather than a structural role.*

*Keywords: infinitesimals, differentials, physics education, university texbooks.*

## INTRODUCTION

The concepts of *infinitesimal* and *differential* are fundamental in physics *problem-solving* that requires calculus. In particular, when using calculus in a physical context, the basic concept that appears in the process of mathematization is that of the *differential*, either in reference to independent variables or as part of differential expressions. As López-Gay and colleagues write, "If we want students to learn how to mathematize real-world problems when [Differential Calculus] is required, a conceptual understanding of this kind of expression and the situations in which these expressions are necessary is essential." (López-Gay et al., 2015).

Research in mathematics and physics education has identified significant challenges students face with the concepts of infinitesimal and differential. Studies suggest that students often develop multiple, context-dependent conceptions of differentials: as an algebraic object in mathematics, or as a pure abstraction or "small amount of something" in physics (Artigue et al., 1990). Despite their procedural use of differential and integral methods, students frequently lack a clear understanding of the conceptual foundations or the conditions that justify these operations, relying instead on superficial cues, such as the term "elementary".

Building on this, López-Gay and colleagues (2015) investigated the conceptions of the differential that students employ in physics, analysing whether these understandings enable the construction of mathematical models or merely support procedural manipulation of pre-given expressions. Their work underscores the polysemic nature of the differential. In mathematics, the differential is primarily concerned with providing a rigorous foundation and formalization of calculus, independent of any physical context. In contrast, in physics, the differential emphasizes the practical application of concepts and reasoning, often prioritizing utility over formal rigor (Artigue et al., 1990, Dunn and Barbanel, 2000)

Our study follows the work of López-Gay and colleagues and aims to analyse how infinitesimals and differentials are treated in university physics textbooks and to investigate if and how the approach to the concept of infinitesimal and differential has changed in physics textbooks over the years. The motivation to investigate the change is due to the impression that physics students, over time, have been encountering more difficulties in activating intuition in dealing with differential equations.

## THE STUDY: AIMS, CONTEXT AND METHODS

For the study, a selection of university physics textbooks has been carried out. The textbooks have been chosen with two criteria: they have to represent the textbooks' evolution over the XX Century; they have to be "classical", authoritative and, then, widely adopted in their historical period in Italy. For the analysis, the chapter on kinematics was chosen as representative of how the approach to the concepts of infinitesimals and differential is used in the book, being typically the first chapter where the students encounter calculus concepts in physics.

In particular, we examined 5 textbooks from the 1940s-1950s (Bernardini, 1942; Castelfranchi, 1945; Perucca, 1941; Rostagni, 1951; Valle, 1947), 4 textbooks from the 1980s-1990s (Bertin et al., 1996; Gettys et al., 1993; Roller & Blum, 1984; Smith & Smith, 1993) and 8 textbooks from the 2000s-2020s (Duò & Taroni, 2021; Focardi et al., 2014; Kleppner & Kolenkow, 2014; Mazzoldi et al., 2021; Mencuccini & Silvestrini, 2016; Resnick et al., 2003; Serway & Jewett, 2023; Sette et al., 2021).

The analysis has been carried out by applying an analytical lens, built on the results of the textbook investigation of López-Gay and colleagues (2015), who pointed out four different possible ways to deal with the differentials in physics textbooks: as a *meaningless* formal element (part of the derivative or integral symbol), as an *infinitesimal increment* ($\mathrm{d}f = \Delta f$ is the infinitesimal variation produced by the infinitesimal variation $\Delta x$), as an *infinitesimal approximation* ($(\Delta f - \mathrm{d}f) / \Delta x$ is an infinitesimal and all the terms $\Delta f$, $\mathrm{d}f$, $\Delta x$ and $\Delta f - \mathrm{d}f$ are also assumed to be infinitesimal numbers) and as a *linear estimate* of the increment ($\lim_{\Delta x \to 0}[(\Delta f - \mathrm{d}f) / \Delta x] = 0$, but $\Delta f$, $\mathrm{d}f$, $\Delta x$ and $\Delta f - \mathrm{d}f$ could have big or small values).

In particular, for each textbook, we first checked whether they contained the ε-δ definition of limit and the term "infinitesimal" with any meaning (infinitesimal number or variable approaching zero). The exposition of the ε-δ definition of the limit is important as it denotes an assimilation of and an adherence to the Cauchy and Weierstrass approach to calculus. The use of the term "infinitesimal" denotes the need to keep a connection with the original setting of Newton and Leibniz.

Then, we classified the concept of differential in the book, following the four categories introduced by López-Gay and colleagues (2015).

Finally, we applied the model of Uhden and colleagues (Uhden et al., 2011) to evaluate whether, in the chapter, the role of mathematics in the text emerged as *structural* or *instrumental*.

## FINDINGS

In Table 1, we report a summary of the findings.

| Year | Author | ε-δ definition of limit | Use of term "infinitesimal" | Differential conception | Role of mathematics |
|---|---|---|---|---|---|
| 1941 | Perucca, E. (1941). *Fisica Generale e sperimentale, vol I, IV ed*, Unione tipografico-editrice torinese, Torino. | No | No | Meaningless | Structural |
| 1942 | Bernardini, G. (1942). *Fisica Sperimentale, parte I, III Ed.* Stabilimento Editoriale Tipo-Litografico V. Ferri, Roma. | No | Yes | Infinitesimal approximation | Structural |
| 1945 | Castelfranchi, G. (1945). *Fisica Sperimentale e Applicata, secondo i più recenti indirizzi, vol. I, IV ed*. Editore Ulrico Hoepli, Milano. | No | Yes | Infinitesimal increment | Technical |
| 1947 | Valle, G. (1947). *Guida alle lezioni di Fisica Sperimentale, parte I, II ed*, Casanova editore, Parma. | No | Yes | Infinitesimal increment | Structural |
| 1951 | Rostagni, Antonio (1951). *Meccanica Termodinamica, vol. I, Fisica Sperimentale*, Libreria universitaria di G. Randi, Padova. | No | Yes | Infinitesimal increment | Structural |
| 1984 | Roller, D. E. & Blum, R. (1984). *Fisica vol. I, Meccanica, Onde, Termodinamica*. Zanichelli, Bologna. | No | Yes | Infinitesimal increment | Technical |
| 1993 | Gettys, W. E., Keller, F. J., & Skove, M. J. (1993). *Fisica classica e moderna, vol. 1, Meccanica e Termodinamica*. McGraw-Hill Italia. | No | No | Meaningless | Technical |
| 1993 | Smith, P., & Smith, R. C. (1993). *Mechanics, II edition*. John Wiley & sons. | No | No | Meaningless | Technical |
| 1996 | Bertin, A., Poli, M., & Vitale, A. (1996). *Fondamenti di Meccanica*. Società editrice Esculapio, Progetto Leonardo, Bologna, | Yes | Yes | Infinitesimal approximation | Technical |
| 2002 | Resnick, R., Halliday, D., & Krane, K. S. (2002). *Physics vol. 1, V edition*. John Wiley & sons. | No | No | Meaningless | Structural |
| 2014 | Kleppner, D., & Kolenkow, R. (2014). *An Introduction to Mechanics, second edition*. Cambridge University Press. | No | Yes (rotation) | Linear estimate | Structural |
| 2014 | Focardi, S., Massa, I., Uguzzoni, A., & Villa, M. (2014). *Fisica Generale, Vol 1, II ed*. Casa Editrice Ambrosiana, Milano. | No | Yes | Linear estimate | Technical |
| 2016 | Mencuccini, C., & Silvestrini, V. (2016). *Fisica, Meccanica e Termodinamica, con esempi ed esecizi*. Casa Editrice Ambrosiana, Milano. | Yes | Yes | Linear estimate | Structural |
| 2021 | Duò, L. & Taroni, P. (2021). *Fisica, Meccanica e Termodinamica*. Edises Edizioni, Napoli | No | Yes | Infinitesimal increment | Structural |
| 2021 | Mazzoldi, P., Nigro, M., & Voci, C. (2021). *Elementi di Fisica, Meccanica e Termodinamica, III ed*. Edises, Napoli. | No | Yes | Meaningless | Technical |
| 2021 | Sette, D., Alippi, A., & Bettucci, A. (2021). *Lezioni di Fisica 1, vol. 1, II ed*. Zanichelli, Bologna. | No | No | Meaningless | Technical |
| 2023 | Serway, R. A., & Jewett, J. W. Jr (2023). *Fisica per Scienze e Ingegneria, vol. I, VI ed.*, Edises, Napoli | No | No | Meaningless | Technical |

**Table 1: Results of the textbooks analysis**

The table shows that, in physics textbooks, the ε-δ definition of limit is rarely introduced and that the term infinitesimal appears to be fundamental over time. In spite of this, the conception of differential tends to become meaningless and this often occurs in the textbooks where mathematics does not have a structural role. In the following section, we provide a possible interpretation of what emerged, to spark the discussion about the specificity of the physical approach to calculus.

**DISCUSSION**

In physical culture, the focal role of measurements leads to preventing any tendency to believe that the information from reality can be "faithfully" represented by *infinitesimal/hyperreal* numerical values, or by *real* numerical values. The more widespread ontological and epistemological view is shaped around the "matter of fact" that every physical measurement is limited by the sensitivity of the measuring instrument and therefore has an intrinsic error. Thus, any physical measure, with any degree of accuracy, can be described by a *rational* number. As noted by Courant and Fritz (1999), thinking of a measurement result as a *real* number is not more than a mathematical *idealization*. "The practical significance of such idealizations lies in the fact that, through the idealizations, analytical expressions become essentially simpler and more manageable". For example, it is simpler and more convenient to work with the notion of instantaneous velocity than with the notion of average velocity. In physics, we feel we have the right to replace a derivative by a difference quotient and vice versa, provided only that the differences are *small enough* to guarantee a sufficiently close approximation. "As long as he keeps knowingly within the limits of accuracy required by the problem, he might even be permitted to speak of the quantities $dx = h$ and $dy = h f'(x)$ as infinitesimals". These "physically infinitesimal" quantities have a precise meaning. They are variables with values which are finite, unequal to zero, and chosen small enough for the given investigation, i.e. smaller than the degree of accuracy required.

Since these "physical infinitesimals" are not truly infinitesimals in the mathematical sense, the problems that plagued infinitesimals in the early developments of mathematical analysis are not encountered for them. Most likely — for the sake of simplicity and manageability — physics teaching is best accomplished using the old and intuitive infinitesimal concepts of Leibniz and Newton.

For these reasons, probably, physics textbooks seem to have been late in receiving the transition from Leibniz and Newton's infinitesimal analysis to the later developments of Cauchy and Weierstrass. As it emerged from our analysis, 20th century textbooks still show clear traces of infinitesimals in Leibniz's vision. The 20th century physics books that we analysed frequently use the term *infinitesimal* and only two of them present the ε-δ definition of limit.

As physics textbooks embrace the change in mathematical analysis, they increasingly marginalize the mathematical definitions of calculus concepts.

Clearly, it is unthinkable in a physics textbook to define the instantaneous velocity of a material body using the ε-δ definition of limit, as in the paradoxical dialogue reported in (Grabiner, 1983):

Student: The car has a speed of 50 miles an hour. What does that mean?

Teacher: Given any $\varepsilon > 0$, there exists a $\delta$ such that if $|t_2 - t_1| < \delta$, then $|(s_2 - s_1)/(t_2 - t_1) - 50| < \varepsilon$.

Student: How in the world did anybody ever think of such an answer?

The ε-δ definition of velocity is too complicated for students to understand.

Physics textbooks rather prefer to consider the concept of limit as a primitive/intuitive concept, without defining its meaning, sometimes referring to calculus textbooks for their understanding. The role of mathematics thus becomes increasingly instrumental and less structural in physics textbooks.

In conclusion, what we observed is that the value of rigor in mathematics led to cutting some conceptual nuances that are, instead, fundamental for physics reasoning. The effects and implications are that mathematics, in physics, is used instrumentally and not for guiding and structuring reasoning.

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# Using an economics model to introduce systems of differential equations

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*An experience in teaching systems of differential equations based on the use of models in the classroom is presented. It is based on modeling and Action, Process, Object, Schema (APOS) theory. In this study, we analyze a modeling situation used in a dynamical systems course for economics students, which needs to be approached with systems of differential equations. This paper contributes to the literature on modeling and the design of a situation that promotes different solution strategies and important information about students' constructions when faced with a new situation where a system of differential equations is needed.*

*Keywords: modeling, APOS theory, differential equations, economics.*

## INTRODUCTION

There are several ways to introduce modeling in the teaching and learning of Calculus. Each approach may contribute to mathematics education by looking at different points of view (e.g. Lesh & Lehrer, 2003; Rasmussen et al., 2012). Those studies addressing natural or social phenomena have called researchers' attention (e.g. Leung et al., 2021). The benefits and limitations of using modeling to promote students' mathematics learning have been studied (e.g. Boaler, 2001; Trigueros & Martínez- Planell, 2024).

Our motivation for using models in this context relates, on the one hand, to the importance of fostering reflection on mathematical concepts and their connection to phenomena in different contexts. On the other, it strengthens students' collaborative work by discussing interesting problems, which fosters the development of new knowledge, ensuring the genuine need for new mathematical concepts.

## THEORETICAL FRAMEWORK

APOS theory is based on Piaget's epistemology. It was developed by Ed Dubinsky (Arnon et al., 2014). It provides tools to evidence students' mathematical knowledge construction. The theory is based on theoretical structures that give it its name: Action, Process, Object, and Schema (APOS) theory. We use Schema structure in this study.

A Schema is a structure consisting of interrelated Actions, Processes, Objects, and other previously constructed Schemas. When students face a problem, they evoke a Schema to solve it. Schemas develop continuously by constructing different types of relationships among their components and assimilating or accommodating new structures. This evolution can be analyzed using three specific levels: Intra-, Inter-, and Trans-, distinguished by the type of relationships among its components a student shows. When Schemas at the Trans- development level are coherent, they can be thematized into an Object on which new Actions can be performed.

Our research questions are: How do students use their economics knowledge when facing a problem involving changes in supply and demand due to inflation? What strategies do they develop in their constructions when working on this situation?

## METHODOLOGY

APOS theory includes its research and teaching methodology: A fundamental element of the APOS theory is a hypothetical model, called a genetic decomposition (GD), that describes the structures and mechanisms necessary to construct a given mathematical concept. Researchers propose this model based on their teaching experience, prior research findings, or historical suggestions. A GD must be experimentally tested and can be validated, refined, or discarded depending on the results. A GD is not unique, different GDs may coexist. The GD is validated when it accounts for the constructions needed for a concept. The GD proposed in Trigueros (2023) is tested in this study.

Although modeling was not originally introduced in this theory, it is compatible with it: When students face a modeling problem, they use previously constructed mathematics, other disciplines, and previous experience Schemas to tackle it. They can define variables and hypotheses that allow them to address the problem. The construction of relations between structures promotes its development and understanding of possible solutions to the proposed modeling problem (Ibid).

Teaching with APOS theory involves using the ACE cycles methodology: activities (A) for students to work collaboratively in teams, whole class discussion (C), and homework exercises (E). The activities used are designed according to the GD. When models are used, the modeling cycle is linked to a specific ACE cycle.

## THE EXPERIENCE

The experience was conducted in a class with 22 students enrolled in the Economics major at a private university where students were taking a dynamical systems course. APOS didactic methodology was used. Students were divided into six small groups of three students and one of four students. They worked in the same group throughout the semester. All their work during class was recorded and collected by the teacher. In this study, we describe the modeling strategies followed by students during class and their work regarding a project done as homework related to the modeling problem.

The teacher took notes of students' work while they discussed the modeling situation and analyzed all the work done at the two modeling sessions and the results of the homework project. Researchers revised students' work and the teachers' notes independently, focusing on students' dialogs and their production during the class, and revised their projects. Researchers then discussed their findings, and only the agreed-upon results were considered in this study. The modeling problem was: At this moment there is inflation in our economy. This has an impact on the economy. Can you predict how the price of a specific good will change in this circumstance?

## SOME RESULTS

We describe the modeling cycles together with ACE cycles. Students started right away to use their economics knowledge. They discussed how inflation would affect the supply and demand of certain goods. Some of them used a specific good such as bread or gas. In three teams a similar discussion took place. Students discussed:

A1: We know that supply and demand are functions, Also, when the price of a good or goods grows, that provokes a growth, an increase of supply.

A2: … producers want to sell more, but demand is decreasing.

A2: what matters now is how the price grows in time, because of inflation, so we need, how the price changes, and then… inflation means price is changing

A1: demand, supply, and price change, but we know that price rate of change is proportional to the difference in price at time *t* and the equilibrium price…

A3: … but we must add inflation; how can we represent that?

Students then decided to suppose that inflation can be a linear, constant, or exponential function, but they considered it linear since it was easier. They also suggested that supply and demand change at the same rate in each period and are equal. Most teams considered inflation as price changes in time and used the price function derivative to consider it. In a whole class discussion, most teams agreed that this situation could be described as: $D = f(p, p')$, $S = g(p, p')$, with $S = D$.

The teacher asked all students to explain what they would do with those equations.

C3: We used $S = c_1 p + c_2 p' + c_3$, that is linear in $p$ and its derivative. Also, demand can be the same, but with other constants such as $a_1$, $a_2$, $a_3$ and as they are equal, it would be something like $p'(t) + k_1 p = k_2$

G2: This considers only supply and demand. What happens with inflation?

G3: It is there in $p'$, is it?

G2: No, this is only because supply and demand change when prices change, and inflation only implies that the price changes due to external factors, and that should be added to the model, but I don't know how.

The teacher invited students to think about how to include the inflation factor in the proposed equations. Students went back to their teams. In this second modeling cycle discussion focused on inflation as a function. In group D:

D3: Inflation makes price change, as they said, we can think of it as a function $I(t)$. It may be $p'(t) = I(t)$ … What do you think?

D1: Well, inflation changes supply and demand, but if we suppose that its impact is stronger on the supply, what will happen?...

D3: we already know that it is proportional, the difference between supply and demand at a time *t* and the equilibrium supply… We had a constant with $S -$

$S_0$. I think that should be negative, like $-c_1S$ and with inflation $I(t)$, the price would change… the change in price has to take into account both of them it would then be something as ... $P'(t)$ could be the sum of both… I cannot explain well, but what I think is that $I(t)$, would be (writes) $P'(t) = -c_1(S(t) - S_0) + F(t)$ and the other the same: $S'(t) = -c_2(P(t) - P_0)$.

D2: So, it looks like a system of equations, but with derivatives. An equation for $P'$ and other for $S'$

D3: Yes, but we have three variables and two unknowns, can we get rid of one?

D1: I see! Yes! If we derive the first equation, we get $P''(t) = -c_1S'(t) + F'(t)$ and substituting $S'$ we have $P''(t) = \mp c_1[c_2(P'(t) - P_0)] + F'(t)$. This has only $P$ but has second derivatives… We should ask the teacher…

In Group G, students started like the other team, but they examined the system of equations $P'(t) = -c_1(S(t) - S_0) + F(t)$ and $S'(t) = -c_2(P(t) - P_0)$. They tried what they used with first-order differential equations (DE):

G4: Each equation is autonomous when we have not added $F(t)$. We can use qualitative analysis. So, we can see that $P' = 0, S(t)$ is constant. And if $S_0$ is zero, $P$ is constant.

G3: I don't understand. Before, we could do a phase line since $P'$ depended on $P$ but here it depends on the other variable….

G1: I don't understand… Maybe we could use two phase lines $P'$ and $S'$ and at $P' = 0$ there is an equilibrium point and in $S' = 0$.

The teacher shared these findings during the discussion with the whole class since they were new for all students. Students explained their work and findings. Many questions were raised regarding how to work with those equations and their meaning. The teacher decided to introduce students to activities designed with the GD. Students reflected on systems of equations and qualitative analysis followed by their equivalence with second order DE and their equivalence. Results showed that students used their previous knowledge and discovered new ideas. Activities designed using the GD complemented the work on the model and created new opportunities for reflection. Students were able to use their economics knowledge and suggested interesting alternatives to work on the model.

## DISCUSSION AND CONCLUSIONS

Students used their economics knowledge to work on the model. They involved changes in supply and demand due to inflation and discovered how to deal with them when change in time was introduced. Students were eventually able to use the model and the new knowledge developed as a means of reflection. The modeling situation helped raise student interest and motivation on systems of differential equations.

Students used two strategies throughout the experience. One of them was the use of supply and demand they had learned in economics. But they were able to find a way to use that knowledge in a dynamic situation. Other interesting strategies were their proposals to deal with the system of DE. Some teams changed the system into a second-order differential equation, and others related their work to previous qualitative analysis while suggesting the possible use of phase lines, which would later open the door to the introduction of the phase plane and its use to analyze the behavior of the system of DE they had found. In terms of students' Schema development, they constructed relations between their economics and differential equations knowledge. They also related it with algebraic systems of equations and autonomous DE qualitative analysis. This Schema kept developing through their work on the model. Key outcomes included the development of the relation between systems of two linear equations and second-order linear DE and the suggestion of the double phase line, which was used to introduce the phase space and the qualitative description of systems of DE.

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# Using Differentials in Thermodynamics

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*Thermodynamics provides a powerful context in which to explore expert and student understanding of partial derivatives, differentials, and chain rules. We summarize here our efforts to identify expert reasoning in this arena and convey it to students as part of the Paradigms in Physics project at Oregon State University, now in its third decade. In particular, we describe expert use of differentials to manipulate partial derivatives and analyze student difficulties in moving between different representations of partial derivatives.*

*Keywords: Teaching and learning of specific topics in calculus, teachers' and students' practices related to calculus across disciplines, differentials, partial derivatives, thermodynamics.*

## INTRODUCTION

A typical question in thermodynamics is to determine the *adiabatic bulk modulus*, defined by $\beta_S = -V\left(\frac{\partial p}{\partial V}\right)_S$, which measures resistance to compression [1]. For example, the entropy $S$ might be given by

$$S = Nk\left(\ln\left[\frac{V}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{5}{2}\right)$$

in terms of the volume $V$ and the temperature $T$, and for an ideal gas we would have the *equation of state* $pV = NkT$, relating the pressure $p$, volume $V$, and temperature $T$; where $\hbar$, $N$, $m$, and $k$ are constants. It is not immediately obvious how to determine the desired partial derivative from the given information, nor which variables are independent—or even how many there are.

We summarize here some of our efforts to investigate understanding of partial derivatives, including a cognitive task analysis of expert approaches to problems such as the one above and thematic analyses of student difficulties in applying their mathematical knowledge to such problems. This work is part of the Paradigms in Physics project at Oregon State University, which for nearly 30 years has reimagined the undergraduate physics major, not only incorporating and adapting modern pedagogical strategies, but also significantly rearranging the content, based on the education research of ourselves and others.

## SOLVING THE PROBLEM WITH DIFFERENTIALS

We begin with a task analysis of the underlying mathematics. The geometric approach to calculus used in the Paradigms project emphasizes infinitesimal reasoning using *differentials* to represent quantities that are "small enough" to model linear differential relationships to the desired accuracy (Dray & Manogue, 2003, 2010; Dray, 2016, Dray et al., 2019). We teach our students to "zap with *d*", converting an equation such as $xy = 1$ to $x\,dy + y\,dx = 0$. This process automatically keeps track of which derivatives have been taken; it is not necessary to decide beforehand which variables are independent.

In the example above, zapping the equation for *S* with *d* and a little algebra yields

$$dS = \frac{3Nk}{2T}dT + \frac{Nk}{V}dV$$

and setting this expression equal to zero (since $S$ is constant) yields a *linear* relation between $dT$ and $dV$, which can of course be solved for either differential. Zapping the ideal gas law with *d* and eliminating $dT$ leads directly to $\beta_S = \frac{5NkT}{3V}$.

## USING CHAIN RULE DIAGRAMS TO KEEP TRACK

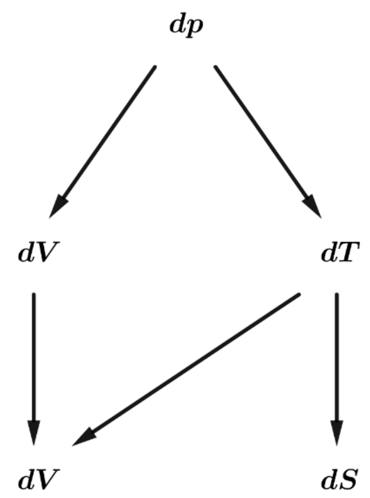


**Figure 1: A chain rule diagram for the calculation of adiabatic bulk modulus. Arrows indicate that the upper differential depends (linearly!) on the lower differential.**

One great advantage of the differentials approach is that it is always possible to solve linear equations involving differentials, unlike the equations for the original variables. One disadvantage of this strategy is that it is easy to lose track of where in the calculation one is. We encourage students to use *chain rule diagrams* such as Figure 1 as a reminder of which partial derivatives are needed. Similar diagrams are often used in mathematics textbooks to represent the multivariable chain rule, although we prefer to write such diagrams directly in terms of differentials. The diagram in Figure 1 is equivalent to the chain rule expression

$$\left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S .$$

## WHAT OUR RESEARCH SHOWED

### Expert Reasoning

Kustusch et al. (2012, 2014) interviewed ten experts in several disciplines while giving them a problem much like the example above. No two experts approached the problem the same way. Three basic strategies, or "epistemic games", were identified using cognitive task analysis: using *substitution* to isolate the independent variable prior to differentiation, using various forms of the multivariable chain rule to relate *partial derivatives,* and using *differentials* to reduce the problem to linear algebra. Substitution was seen to be problematic if the given equations are difficult to solve, and partial derivatives and differentials were described by one expert as encoding the same information quite differently. Interestingly, some of the experts mentioned their concern that some of the (correct) moves that they made would not be considered "legal" by mathematicians, especially in the context of working with differentials. This research prompted the development of the "Partial Derivatives Machine" (PDM), a simple mechanical device with springs and pulleys that provides an exact mathematical analogue to classical thermodynamic systems such as gas in pistons. We have developed curriculum around the PDM and done some initial studies on its effectiveness in the classroom (Roundy et al., 2015; Paradigms Team, 2015–2024).

### Student Reasoning

Founds et al. (2017) used an emergent coding scheme to identify and categorize the solution methods of physics students in the Paradigms program on analogous chain rule problems from both a pure algebra question on a quiz (N=29) and then a thermodynamics inspired question that was part of the final exam (N=27). The study examined both what solution strategies students chose to employ and what types of conceptual errors they made. This analysis used a more finely grained classification of the solution strategies, namely *variable substitution, differential substitution, implicit differentiation, differential division, chain rule diagrams,* as well as several strategies that did not lead to an answer. Each of these strategies except the first had been explicitly discussed in class, with emphasis on differential substitution and chain rule diagrams and the quiz was reviewed in class before the final exam.

Many students (31%) attempted to use the familiar technique of variable substitution on the (much easier) quiz problem, where it was indeed a viable solution strategy, but fewer (only 11%) on the exam. It is not clear whether students recognized that the exam computation would be extremely lengthy, or whether they had mastered another technique. No students made conceptual errors with this technique. Differential substitution was used by more students on the exam (44%) than on the quiz (21%) and by the time of the exam no students made conceptual errors with this method. Chain rule diagrams were also relatively common, (21%) on the quiz and (22%) on the final. While several students made conceptual errors in their chain rule diagrams on the quiz (building incorrect diagrams and/or misreading them), none made such errors on the exam. Other methods were less common.

In a follow-up study of 12 students, Founds & Manogue (2022) found, using thematic analysis, that the difficulties many junior-level physics students experience may be related to their unfamiliarity with Leibniz notation. In addition, this study showed that these students do not know to eliminate extra dependent variables in systems of equations.

### Multiple Representations

In a separate study, Bajracharya et al. (2019) asked eight student interviewees a more difficult prompt: to determine a particular partial derivative from data with some presented graphically and other data presented numerically in a table. To solve this problem successfully, interviewees not only needed to derive a chain rule analogous to the one above, but also to identify which partial derivative could be found from the data as presented. This study introduced *representational transformation diagrams* as a method to describe student problem-solving strategies. These strategies included both graphical analysis and analytic derivations using tools such as differentials and tree diagrams. The analysis focused on students' ability to transform one representation into another, identifying several classes of transformations such as *translation*, *consolidation*, and *dissociation*. Consolidation, a process in which a student transforms two or more representations into a single representation, and dissociation, a process in which one representation is expanded into two or more representations, were found to be the most common places for interviewees to encounter difficulties.

## SUMMARY

These results document the difficulties some students have based on their mathematical training when trying to master the expert reasoning around (partial) derivatives used in physics. They also document the existence of several expert approaches to the same task, both across disciplines and within a single discipline. Helping students become experts will require interdisciplinary coordination.

## NOTES

1. This common generalization of Leibniz notation for partial derivatives in thermodynamics will be unfamiliar to many mathematicians and most students, with the subscript indicating the derivative *with the entropy S held fixed*.

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# Posters

# A history-based artifact to mediate calculus contents

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*Keywords: artifacts, tangent, derivative/antiderivative, Theory of Semiotic Mediation.*

## A HISTORY-BASED ARTIFACT

This paper proposes incorporating history and historical artifacts into hands-on activities on infinitesimal calculus, aligning with Italian research on workshop activities involving "mathematical machines" (Bartolini Bussi & Maschietto, 2006). We have utilized a new device rooted in historical concepts related to the mechanical implementation of solutions for inverse tangent problems (Bos, 1988). Beyond their historical importance in the development of calculus, these ideas maintain a strong connection with material implementations, implying the creation of scientific instruments for demonstration, education, and practical application (Tournès, 2009).

Our artifact (Fig. 1), realized with FabLab tools (https://www.machines4math.com/), can be assembled in various configurations (Crippa & Milici, 2023): to introduce the concept of the tangent and inverse tangent problems (https://www.youtube.com/watch?v=LMLt90R8zHA), to trace exponential and parabola (https://www.youtube.com/watch?v=kqtU9GpcN78), and to construct derivatives and antiderivatives (https://www.youtube.com/watch?v=TyxCAR317HE).

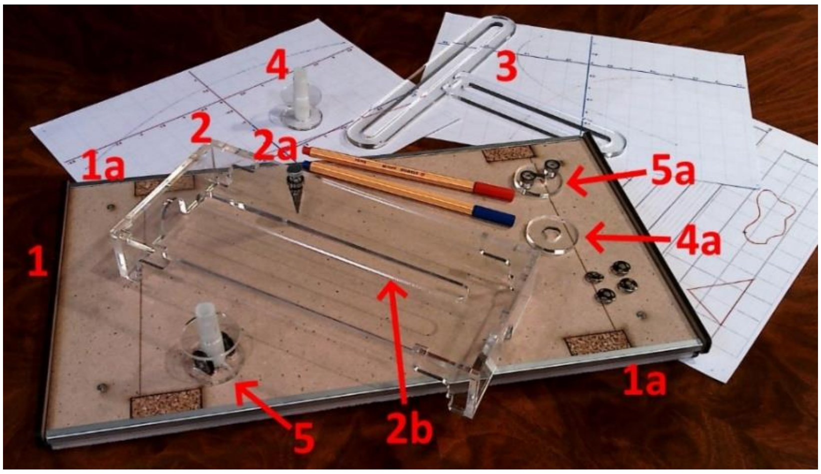


**Figure 1. Components of the artifact.**

## THEORETICAL BACKGROUND

This study examines our new artifact from a semiotic perspective, emphasizing the crucial role of signs and representations in mathematics. The main theoretical component is the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008); it applies a Vygotskian perspective to mathematics education, suggesting that teachers use specific artifacts to mediate mathematical meanings. A relevant theoretical tool is the analysis of the semiotic potential of an artifact, which is crucial for designing student tasks and guiding teacher actions.

## FIRST RESULTS AND FURTHER PERSPECTIVES

Our in-progress research project aims to evaluate our artifact for teaching calculus, focusing on its use, constraints, and manipulation to create effective tasks. We analyzed how a secondary school teacher explores this artifact from a semiotic perspective (Maschietto & Milici, 2024) and we identified different configurations and crucial elements for task design by analyzing its semiotic potential. In its most straightforward configuration, our artifact offers a material representation of the tangent line; in other configurations, the tangent is mechanically guided to generate curves.

We will start to test tasks to bring out the meanings embedded in the artifact. On the one hand, we plan to explore the artifact to consolidate mathematical meanings with university students. On the other hand, we are designing a teaching experiment to mediate calculus meanings; it will be carried out with high secondary school students. Additionally, the historical and epistemological aspects are essential for maintaining student interest: the artifact allows the integration of historical experiences and original manuscripts to enhance learning.

## ACKNOWLEDGMENTS

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# A preliminary analysis of realistic problems concerning derivatives in a widely-adopted secondary textbook

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Calculus enjoys a unique position in mathematics education, serving as a foundational tool across numerous scientific and technical disciplines. Yet, despite its inherent interdisciplinary nature and pervasive applications across diverse fields, calculus appears to remain typically taught as a self-contained mathematical subject with limited explicit integration into other fields (Biza et al., 2022).

The problem of the widely accepted applicability and interdisciplinarity of calculus and its concomitant high intra-disciplinary mode of institutional teaching is therefore very deep and crucially involves also how the discipline is explicitly and implicitly legitimized in terms of its articulated applications. As to implicit institutional legitimation, starting from the pioneering socio-critical work of Dowling (2000), various authors have argued how mathematical activity in general and problem solving in particular implicitly serves to propagate messages concerning the applicability and the connected legitimation of mathematical knowledge. Indeed, engagement with mathematical exercises and, in particular, with realistic problem solving, according to Lundin (2012), serves to deliver various messages concerning the relationship between mathematics and its domain(s) of application. Therefore, the very engagement with mathematical practice and particularly with problem solving constitutes one of the ways in which engagers are institutionally taught about the utility and relevance of mathematical concepts in various extra-mathematical domains.

Aiming to tackle an aspect of this problem with particular reference to calculus as introduced in secondary education, I will furnish in this presentation a preliminary analysis of the realistic problems/exercises concerning derivatives taken from a widely adopted secondary school textbook in Italy. In other words, I will present a preliminary exploration of the following question: What is the typical image of the extra-mathematical applicability of the mathematical concept of derivative presented in this textbook?

The textbook chosen for this ongoing study (i.e., Bergamini et al., 2017) was published by one of the oldest and largest Italian educational publishing houses. The book is considered to be one of the most adopted book for secondary school mathematics, according to relatively recent reports (Montanari, 2019). This book is usually adopted in the final year of high school when students are first introduced to calculus. Given the structure of the Italian educational system this can be thought to be, for many Italian students, the first exposure to calculus before they enroll in mathematical analysis courses at university. The book is organized in chapters which revolve around one fundamental mathematical concept and contains a copious amount of exercises or problems. Some of these are manifestly labelled in the book with the marker "Realtà e modelli" (i.e., "Reality and models", my translation), thus constituting an explicit way to signal to the reader that these problems or exercises, in particular, show the applicability of the learnt concepts to reality and to models of an extra-mathematical nature. For this reason, these problems can be argued to constitute a privileged way to access the book authors' intended idea of application of the concepts of mathematics outside of it.

Thus, in this presentation, after exhibiting general data on the overall amount and relative proportions of the exercises/problems in the selected chapter, I will offer a preliminary analysis of the aforementioned group of marked problems and illustrate the degree of realism that these manifest by presenting two selected problem which can considered to be typical in this respect. Overall, I will argue that, generally speaking, these epitomize a phenomenon that, following Dowling (2000), can be considered that of a simple "disguise" of traditional mathematical exercises, i.e., the rewording of a traditional mathematical problem with superficial extra-mathematical features in order to make it appear more "realistic".

Moreover, I will attempt to nuance how engagement with these and similar problems in secondary school is connected to the interiorization of a message about the applicability of calculus to reality. I will finally discuss the potentially far-reaching institutional consequences which can be argued to generally affect the epistemologies acquired by students concerning the extra-mathematical applicability of calculus and of mathematical analysis also in tertiary education (cf. Satanassi et al., 2022).

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# Developing aspects of the wave function in quantum mechanics through analogical activity: A textbook analysis

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*Key words: Quantum mechanics, analogy, inner product, integral, probability.*

## INTRODUCTION

Quantum mechanics is deeply intertwined with mathematics. In our research, we investigate how linear algebra concepts are leveraged, symbolized, and understood in quantum mechanics. For example, spin and energy are both quantized phenomena that can be mathematized via discrete, finite-dimensional linear algebra. The physical observable of position, however, is mathematized via continuous, infinite-dimensional functional analysis. Associated computations differ among the finite and infinite, such as the inner product as a finite sum versus as a definite integral. How might a shift from discrete to continuous be explained mathematically and conceptually in a course text? In this poster, we study the research question: *How does a quantum mechanics textbook employ analogizing activity to support its transition from the discrete contexts of spin and energy to the continuous context of position?* This research lives at the intersection of not just mathematics and quantum mechanics, but also of linear algebra and calculus.

## THEORETICAL FRAMEWORK

As Ryder et al. (2023) state, "Analogies are known to be powerful tools for making sense of unfamiliar ideas in terms of already understood concepts" (p. 284) because they help us use knowledge from one domain to develop new knowledge in another (Serbin & Wawro, 2024). Even textbook writers (e.g., McIntyre, 2012) at times employ analogy to model quantum systems and make them more familiar and understandable (Ryder et al., 2023). Gentner (1983) described *analogy* as a mapping from a *source domain* to a *target domain* that preserves relations between objects and their attributes in the source and target. Glynn (1989) analyzed elementary through university science textbooks and found six analogical operations, such as cue source retrieval, map similarities between source and target, and indicate where the analogy breaks down. Serbin and Wawro (2024) investigated pedagogical moves for analogical activity that could support students in developing an understanding of quantum eigenequations $\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle$, $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$ as instantiations of a central concept across the contexts of spin, energy, and position, respectively. We identified eight additional analogical activities, such as juxtaposing symbols representing objects in the source and target, and highlighting sameness of aspects in the source and target.

## METHODS

In this poster, we will analyse a quantum mechanics textbook by McIntyre (2012), focusing on the introduction of the wave function (section 5.3, pp. 112–119). We chose this because it serves as the transition from discrete to continuous for the quantum state vector, associated representations, and computations such as probability amplitudes. In our analysis of analogical activity, we will first look for parallel sentence structure and words commonly used to indicate analogy, such as is/are, like/same/similar, or different/difference (Ryder et al., 2023). We will also identify the source and target domains (Gentner, 1983), as well as broader analogical activity categories consistent with the literature (e.g., Glynn, 1989; Ryder et al., 2023; Serbin & Wawro, 2024).

## BRIEF ANALYSIS EXAMPLE

The textbook section begins: "To better understand the new concept of a wave function $\psi(x)$, let's see how it relates to the quantum state vector $|\psi\rangle$ we used in spins" (p. 112). The source and target domains are spin and position, respectively, the mapped objects are $|\psi\rangle$ and $\psi(x)$, and the analogical activity includes Introducing Target Concept, and Juxtaposing Symbols Representing Source and Target Objects. Later the text states "*all values of position $x$ are allowed*. This is in stark contrast to the case of the spin component $S_z$ … the spectrum of eigenvalues of position is *continuous*, and the spectrum of eigenvalues of spin is *discrete*" (p. 13, emphasis in original). The mapped object is the eigenvalue spectrum with the attribute of discrete versus continuous, with the activity of Indicating where the Analogy Breaks Down. Finally, the text focuses on the required sum of all possible probabilities: "In the discrete spins case this meant that $\sum_{\pm}\mathcal{P}_{\pm} = \sum_{\pm}|\langle\pm|\psi\rangle|^2 = 1$ … However, because the spectrum of position eigenvalues is continuous rather than discrete, the sum over discrete probabilities must be changed to an integral over the continuous probability function…thus the normalization condition is $\int_{-\infty}^{\infty}\mathcal{P}(x)dx = \int_{-\infty}^{\infty}|\psi(x)|^2dx = 1$" (pp. 114–15). Focusing on discrete versus continuous attribute, Highlighting Sameness for the total probability sum, and Juxtaposing Symbols facilitated introducing the integral as an inner product and norm condition.

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# Enhancing Understanding of Integrals in Physics and Mathematics Education: A revision of the Test of Calculus and Vectors in Mathematics and Physics

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## INTRODUCTION AND THEORETICAL FRAMEWORK

Physics heavily relies on mathematics, yet proficiency in procedures does not ensure conceptual transfer across contexts (Uhden et al., 2012; Redish, 2005). This disconnect is especially evident in early physics courses. Mathematics plays a structural role in physics reasoning, shaping conceptual understanding (Pospiech, 2019). A key challenge is interpreting definite integrals, especially in relation to the Fundamental Theorem of Calculus (Bajracharya, 2014). Jones (2013, 2015a) identifies three symbolic forms corresponding to different conceptualizations of the integral: area, anti-derivative, Riemann sum. The Riemann sum approach is essential in physics but often underutilized by students (Jones, 2015b). The *Test of Calculus and Vectors in Mathematics and Physics* (TCV-MP; Carli et al., 2020) is a multiple-choice test administered at the start of Physics 1 (after Calculus 1) courses in Science and Engineering programs at our institution. It includes paired mathematics and physics items on integrals derivatives, and vectors. This contribution presents the revision of the integrals section to improve balance among sections, enhance item quality, and incorporate recent research. The revision involved analyzing previous test results from 2018 to 2024, 13 instructor interviews, and 11 think-aloud student interviews. Items were initially developed in open-ended form, and the outcomes of the interviews were then used to construct the distractors for the multiple-choice version.

## ITEM REVISION AND PRELIMINARY FINDINGS

The revised items cover all three conceptualizations while addressing physics-specific challenges like spatial variation, differentials, and non-uniformity.

| Item pair | | Topic | | Conceptualization | Representation | |
|---|---|---|---|---|---|---|
| Math | Phys | Mathematics | Physics | | input | output |
| 7M* | 7P | $f' \to \Delta f$ (area + FTC) | velocity → displacement | Area under the curve | graphs | words |
| 8M* | 8P* | $f \to F$ (integral funct.) | velocity → displacement | Signed area under the curve or anti-derivative | graphs | graphs |
| 9M* | 9P* | $f' \to f$ (area + FTC) | velocity → position | Signed area under the curve | graphs | numbers |
| Sum M | Sum P | accumulation of area | linear density → mass | Riemann sum | words formal | words |
| Anti M | Anti P | $f' \to f$ (antiderivative) | acceleration → velocity | Anti-derivative | words formal | numbers |

**Table 1: Categorization of the new and modified items for the revised TCV-MP.**

The Riemann sum form was introduced through a new item on linear density (Fig. 1). Interviews revealed a variety of interpretations of ρ(x)dx, with only a few students identifying it as an infinitesimal mass element.

**Item Sum M.** Consider a positive (real-valued) function $f(x)$ defined on $\mathbb{R}$ and the expression $\int_0^2 f(x)\,dx$.
What do $f(x)\,dx$ and the whole expression $\int_0^2 f(x)\,dx$ respectively represent (geometrically)?

**Item Sum P.** The linear density of a bar of length $L$ is given as a function of the distance $x$ from one end of the bar by $\rho(x)$. Consider the expression: $\int_0^L \rho(x)\,dx$.
What do $\rho(x)\,dx$ and the whole expression $\int_0^L \rho(x)\,dx$ respectively represent (physically)?

**Fig.1: The new item pair on the 'adding up pieces' conceptualization of the integral.**

The inclusion of new items and modification of the existing ones were designed to cover all three conceptualizations while keeping a manageable test length. The revised integral section of the TCV-MP expands from three to five item pairs, offering broader coverage of graphical and formal aspects, and addressing specific conceptual difficulties (e.g., signed area). Interview analysis supported students' over-reliance on procedural approaches, even when accumulation reasoning would be more appropriate. The revised test is currently being validated in its multiple-choice format.

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# Derivative. Meanings in filling containers

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*Keywords: up to five keywords divided by commas, used dot at the end.*

## INTRODUCCTION

Derivative is fundamental concept in mathematics since facilitates the construction and understanding of concepts such as the tangent line and in other disciplines allows modeling and studying phenomena related to the rate of change. Its importance is such that a large space is dedicated to it in the calculus curriculum, especially the modeling, like Gay & Jones (2008) they say: We recognize that experience modeling has relevance in the sense of participation and control in solution processes. Modeling requires understanding the meaning of concept, its interpretations, use of some derivation techniques and solution of simple differential equations. The didactic proposal that we present finds one of its rationales there, in this case the filling of containers. We begin analysis in simple familiar containers where it is possible to imagine the solution using purely intuition and some qualitative elements; After the analysis focuses on containers more complex whose design requires to build non-routine functions like functions defined by pieces, where they show relationship between meanings and interpretations of the derivative, like the rate of change and its definition, which as Leinhardt, *et.al.* (1990, pp. 1-64) points out: is not at all easy since it requires generating new parts that are not given, in addition to the fact that the representation of mathematical relationships using variables is a powerful but difficult process for students to learn (Kieren, 1992, pp. 390-419).

## THE EXPERIMENT

### Filling two vertical cylinders with different spokes

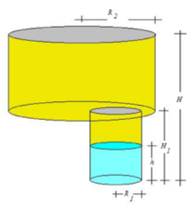

**Figure 1**

The volume of container is:

$$V(h) = \begin{cases} \pi R_1^2 h & si\ 0 \leq h \leq H_1 \\ \pi R_2^2 h & si\ H_1 < h \leq H \end{cases}$$

Using the chain rule, we get:

$$\frac{dV}{dt} = G = \begin{cases} \pi R_1^2 \frac{dh}{dt} \\ \pi R_2^2 \frac{dh}{dt} \end{cases} \quad \text{y} \quad \frac{dh}{dt} = \begin{cases} \frac{G}{\pi R_1^2} & si\ 0 \leq h \leq H_1 \\ \frac{G}{\pi R_2^2} & si\ H_1 < h \leq H \end{cases}$$

Its graph is:

**Figure 2**

Solving the differential equation, we have that the height function is given by:

$$h(t) = \begin{cases} \dfrac{G}{\pi R_1^2} t + C_1 & si\ 0 \leq t \leq T_1 \\ \dfrac{G}{\pi R_2^2} t + C_2 & si\ T_1 < t \leq T \end{cases}$$

Where $T_1$ and $T$ are the times, it takes for the lower cylinder and the complete container to fill respectively. Since *h(0)=0*, then $C_1 = 0$. Also, as $h(T_1) = H_1$ then $T_1 = \frac{\pi R_1^2 H_1}{G}$ , we get $C_2 = H_1\left(1 - \frac{R_1^2}{R_2^2}\right)$. Substituting we get the expression:

$$h(t) = \begin{cases} \dfrac{G}{\pi R_1^2} t & si\ 0 \leq t \leq \dfrac{\pi R_1^2 H_1}{G} \\ \dfrac{G}{\pi R_2^2} t + H_1\left(1 - \dfrac{R_1^2}{R_2^2}\right) & si\ \dfrac{\pi R_1^2 H_1}{G} < t \leq T \end{cases}$$

To calculate the total filling time, it's enough done *h(T)=H* in the last expression and we get:

$$T = \frac{\pi}{G}\left(HR_2^2 + H_1(R_1^2 - R_2^2)\right)$$

We must say that this proposal addresses the general problem of filling container.

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# The role of calculus in advanced physics courses for teachers – instructors' views

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## INTRODUCTION

It is generally agreed that "doing physics" is impossible without a sound calculus background: the fundamental relations governing the dynamics of physical processes are differential equations. Part of the challenge of learning and teaching physics is its relation to mathematics. At the undergraduate level, Hitier and González-Martín (2022) have highlighted how the (inconsistent) use of derivatives in calculus and mechanics courses, taken simultaneously, leads to student difficulties. More generally, Biza et al. (2022) highlighted the different roles that calculus courses play for physics and engineering degree programs. Before choosing their specialization, however, students usually learn physics by interacting with schoolteachers. Naturally, a physics teacher should be proficient in both mathematics and physics (Shulman, 1986). Therefore, physics teacher training draws on both disciplines (Pospiech et al., 2019), to varying degree. For example, Eylon et al. (2010) studied how the instructor of a quantum mechanics course for teachers reduced the course's mathematical load, focusing on developing a qualitative "sense of understanding" of quantum mechanical principles. The last example highlights not only the different roles that mathematics plays in teacher training, but also the relevant considerations that an instructor employs when teaching.

## METHODOLOGY

There are several opportunities in Israel for in-service physics teacher professional development: few-hour workshops, professional learning communities, and MSc/PhD degrees in science education. The Rothschild-Weizmann (RW) Program for Excellence in Science Teaching at the Weizmann Institute of Science is an MSc program that combines courses about pedagogy as well as discipline-specific advanced content courses. The program aims to enhance teachers' confidence and motivation by providing opportunities to advance their knowledge and participate in professional development activities. In the physics branch of the program, prior to the beginning of their first semester, the teachers attend an intense two-day workshop focusing on differential and integral calculus. The workshop instructor continues to teach a mathematics course during the first semester. Evidently, the RW program aims for its graduates to be in command of calculus and understand its relevance for physics. The research question we set to study therefore is: *What do the instructors of the RW program wish to convey to teachers about the role of calculus in physics, and what considerations affect their teaching?*

The study will involve interviews with the current physics instructors of the program.

## PRELIMINARY RESULTS AND OUTLOOK

So far we have interviewed three instructors (note, the mathematics instructor was unavailable before submission). Our preliminary results indicate that several profiles emerge. One wishing to convey a coherent picture of physics based on differential equations, contrary to the segmented secondary school curriculum,

> "I wanted to reach Maxwell's equations so they (the teachers) see how all of these…things, that in school are completely separate…how the entire picture looks."

A second profile emphasizes physical reasoning over equations, because qualitative sense-making has a better chance of reaching the classroom:

> "I stop myself from dealing with the mathematical formalism…I find it important that they demonstrate…that conceptual understanding…because it has a chance of passing on to their students. The formulas have no chance."

A third profile values both physics and mathematics knowledge, choosing subjects according to the teachers' level of mathematics:

> "There is so much to know… there's a lot of value in that… (I learned to) avoid things that are too abstract or mathematical."

The instructors convey to the teachers different roles of calculus in physics, rooted in their commitment to the discipline, to science in general, or to students, exposing a gap between the program's goal and the way the instructors employ it in their teaching.

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# Utilizing an AI Math Bot in Undergraduate Mathematics Classrooms to Explore Students Learning Calculus

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## INTRODUCTION

Generative Artificial Intelligence (Gen AI) is transforming the landscape of students who are learning calculus through a traditional lens. In 2024, the National Council of Teachers of Mathematics (NCTM) urged mathematics teachers to implement AI in classroom instruction in a manner such that students engage with AI tools without replacing human interaction. An example of a Gen AI tool includes ChatGPT, an AI that can be utilized as a teaching assistant to aid students in personalized learning, which includes offering feedback on assessments (Lo, 2022). In the course entitled Calculus I for Science and Engineering, Dr Abby Williams, Assistant Teaching Professor of Mathematics at Northeastern University, created an AI Math Bot which she named Ada to support students' learning of calculus beyond the classroom. Ada is a customized version of ChatGPT that is tailored to students' personalized learning of calculus content by using examples from a calculus textbook that is built into it. In addition, Ada generates practice examinations that mock past examination papers and provide feedback on students' work.

To explore the potential impact of Ada in a calculus classroom, Wenger's (1998) communities of practice lens is applied as the overarching theoretical framework. Wenger (1998) describes the characteristics of a community of practice as people coming together in sharing a common goal. The practice in this study involves students enrolled in Calculus for Business who are engaging in practice with Gen AI. In Wenger's (1998) work, he refers to an artifact as a tool that can be used as a vehicle to solve problems in the practice. In this instance, Gen AI tools such as Ada are examples of artifacts that help calculus students understand and solve calculus applications. Narrative inquiry (Clandinin, 2022) is the methodological approach employed as it enables insight into students' written reflections on using Ada as a teaching assistant. The guiding research questions for this study include:

1. How does Ada aid students' learning of calculus beyond the classroom?
2. How do calculus students benefit from using Ada as a teaching assistant?
3. What is the potential impact of students' performance in calculus when utilizing Ada as a teaching assistant?

## METHODS

Participants for this pilot study included first-year university students enrolled in a Calculus for Business module. These students are non-mathematics specialists and are majoring in fields such as social sciences, humanities, and business. The design of the activity included a worksheet that has a set of seven calculus problems. This was used as a practice examination to prepare the students for their final examination. During the activity, calculus students worked on problems individually, and the instructor reviewed each problem with the entire class. The students uploaded their work on each problem to CANVAS. In addition, the students were encouraged to write a reflection of Ada's responses and the use of Ada as a teaching assistant. Data collection includes students' written work, both instructor and Ada solutions, and students' optional reflections. Data analysis entails reviewing students' reflections and written work along with Ada and instructor solutions.

## DISCUSSION

In this pilot study, calculus students voluntarily shared their work on CANVAS to show how they solved the seven-question final review worksheet. The notable difference came from questions regarding integration by parts and compound interest questions. Calculus students followed the instructor's approach of setting up a box to determine the variables 'u','du','v','dv'. In contrast, Ada just mentions that step # 1 is integration by parts

On questions involving compound interest, Ada had the correct step-by-step approach in solving compound interest questions; however, Ada computed the wrong values. This was a teaching moment, however, wherein the instructor could ask students to check Ada's computations to see if they were correct by using a calculator. While the data collection came from a small sample, individual reflection pieces by each student would have been useful in understanding how Ada could be fully utilised as an AI teaching assistant with students learning calculus. I hypothesise that this pilot study will contribute to practitioners in the field in understanding how Gen AI could be a useful AI teaching assistant tool for students to learn calculus.

## REFERENCES


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